Algebraic topology - Homework 1

November 06, 2024

($\star$) = not for submission, but make sure you understand how to do it
($\star\star$) = not for submission, a bonus question which I find interesting

Questions (1) and (2) are not for submission, but they are good for some experience with category theory.

(1) ($\star$) **Isomorphisms.** Let $\mathscr{C}$ and $\mathscr{D}$ be categories.

(a) Let $F\colon \mathscr{C} \to \mathscr{D}$ be a functor and $f\colon X \to Y \in \mathscr{C}$ a morphism. Show that if f is an isomorphism, then $F(f)$ is also an isomorphism.

(b) Let $F, G\colon \mathscr{C} \to \mathscr{D}$ be functors and $\alpha\colon F \to G$ a natural transformation. Show that if α is an isomorphism in $\mathrm{Fun}(\mathscr{C}, \mathscr{D})$, then $\alpha_X\colon F(X) \to G(X)$ is an isomorphism for every X.

(c) Show that the converse of Exercise 1b also holds: if $\alpha_X\colon F(X) \to G(X)$ is an isomorphism for every X, then α is an isomorphism in $\mathrm{Fun}(\mathscr{C}, \mathscr{D})$. Such natural transformations are called *natural isomorphisms*.

(2) ($\star$) **Classifying categories of groups.**

(a) Let G be a group. Define the category $\mathrm{B}G$ as having one object $*$, with $\mathrm{Hom}_{\mathrm{B}G}(*, *) = G$ and composition is given by group multiplication. Show that for two groups G, H functors $\mathrm{B}G \to \mathrm{B}H$ are in bijection with group homomorphisms $G \to H$.

(b) Let G, H be groups. Let $\varphi, \psi\colon G \to H$ be two homomorphisms (thought of as functors $\mathrm{B}G \to \mathrm{B}H$). What are the natural transformations $\varphi \to \psi$?

(c) Give an explicit description of functors $\mathrm{B}G \to \mathrm{Set}$, and of natural transformations between two such functors.

(3) **Semisimplicial sets.** In the following question we will show that the description of semisimplicial sets from the lecture agrees with the categorical one, based on presenting the category $\mathrm{s}\boldsymbol{\Delta}$ with "generators and relations":

(a) Let $\delta_i^{[n]}\colon [n-1] \to [n]$, $0 \le i \le n$ be the strictly increasing function defined by skipping i. Prove that any strictly increasing $\alpha\colon [m] \to [n]$ admits a unique decomposition:

$$\alpha = \delta_{i_k}^{[n]} \circ \delta_{i_{k-1}}^{[n-1]} \circ \cdots \circ \delta_{i_2}^{[n-k+2]} \circ \delta_{i_1}^{[n-k+1]}$$

with $i_1 < i_2 < \cdots < i_k$.

(b) Prove that the category $\mathrm{Set}_{\mathrm{s}\boldsymbol{\Delta}} := \mathrm{Fun}(\mathrm{s}\boldsymbol{\Delta}^{\mathrm{op}}, \mathrm{Set})$ is isomorphic to the category SSS defined in class.[1] Namely, exhibit a bijection between the objects and the morphisms of the two categories that respects compositions and identity.

(c) Given $X \in \mathrm{Set}_{\mathrm{s}\boldsymbol{\Delta}}$, prove that the geometric realization $|X|$ is homeomorphic to the topological space

$$\bigsqcup_{n\geq 0}(X_n \times \Delta^n) \big/ (\sigma, \alpha(t)) \sim (\alpha^*\sigma, t)$$

where $\sigma \in X_n$, $\alpha\colon [k] \to [n]$, and $t = (t_0, \ldots, t_k) \in \Delta^k$, where we define

$$\alpha(t_0, \ldots, t_k) = t_0 e_{\alpha(0)} + \cdots + t_k e_{\alpha(k)}$$

and the function α^* is the value of the functor X on the morphism α.

(d) Given $X \in \mathrm{Set}_{\mathrm{s}\boldsymbol{\Delta}}$, prove that the semisimplicial complex $C_\bullet^\Delta(X)$ is a chain complex. Namely, show that $\partial_n \circ \partial_{n+1} = 0$.

(4) **Functors.**

(a) Show that $|-|\colon \mathrm{Set}_{\mathrm{s}\boldsymbol{\Delta}} \to \mathrm{Top}$ extends to a functor.

(b) Show that $\mathrm{Sing}\colon \mathrm{Top} \to \mathrm{Set}_{\mathrm{s}\boldsymbol{\Delta}}$ extends to a functor.

(c) For every $X \in \mathrm{Set}_{\mathrm{s}\boldsymbol{\Delta}}$ and $Y \in \mathrm{Top}$, construct a bijection

$$\hom_{\mathrm{Top}}(|X|, Y) \simeq \hom_{\mathrm{Set}_{\mathrm{s}\boldsymbol{\Delta}}}(X, \mathrm{Sing}(Y))$$

(d) ($\star$) Contemplate the fact that this bijection is natural in X and Y. This exhibits $|-|$ and Sing as *adjoint functors*.

(5) **Simplicial Homology.**

(a) For $n \geq 0$, let $\vec{\Delta}^n \in \mathrm{Set}_{\mathrm{s}\boldsymbol{\Delta}}$ denote the following semisimplicial set:

- For $k \geq 0$, the k-simplicies are $\vec{\Delta}^n_k = \hom_{\mathrm{s}\boldsymbol{\Delta}}([k], [n])$
- For $\alpha\colon [k] \to [k'] \in \mathrm{s}\boldsymbol{\Delta}$, the corresponding function $\alpha^*\colon \vec{\Delta}^n_{k'} \to \vec{\Delta}^n_k$ is given by

$$\begin{aligned} \alpha^*\colon \hom_{\mathrm{s}\boldsymbol{\Delta}}([k'], [n]) &\to \hom_{\mathrm{s}\boldsymbol{\Delta}}([k], [n]) \\ \beta &\mapsto \beta \circ \alpha \end{aligned}$$

Prove that $|\vec{\Delta}^n|$ is homeomorphic to Δ^n and compute the simplicial homology of $\vec{\Delta}^n$.

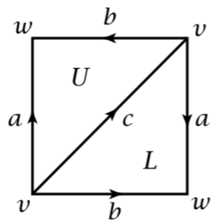
(b) Let $\partial\vec{\Delta}^n$ denote the same semisimplicial set as $\vec{\Delta}^n$, except we remove a single n-simplex

$$\partial\vec{\Delta}^n_n = \hom_{\mathrm{s}\boldsymbol{\Delta}}([n], [n]) \setminus \{\mathrm{id}_{[n]}\}.$$

Prove that $|\partial\vec{\Delta}^n|$ is homeomorphic to S^n, and compute the simplicial homology of $\partial\vec{\Delta}^n$.

(c) Consider the semisimplicial set $X \in \mathrm{Set}_{\mathrm{s}\boldsymbol{\Delta}}$ defined by the diagram below

[1] Generally the more correct notion is equivalence of categories, but in this case they are even isomorphic



with the two horizontal and vertical lines identified according to arrows. Prove that $|X|$ is homeomorphic to the *real projective plane* $\mathbb{RP}^2 := {}^{S^2}/_{-v\sim v}$, and compute the simplicial homology of X.

(d) ($\star\star$) Define a semisimplicial space whose geometric realization is homeomorphic to $\mathbb{RP}^n$, and compute its homology.

(6) **Homotopy.**

(a) Construct the category hTop whose objects are topological spaces and whose morphisms are equivalence classes of continuous maps up to homotopy. Namely, you need to show that composition is well-defined.

(b) Define explicitly a homotopy equivalence between the following two subspaces of $\mathbb{R}^2$:

$$X = S^1 \cup (\{0\} \times [-1,1]) \qquad Y = (S^1 + (-2,0)) \cup ([-1,1] \times \{0\}) \cup (S^1 + (2,0))$$