Algebraic topology - Homework 2

November 13, 2024

($\star$) = not for submission, but make sure you understand how to do it
($\star\star$) = not for submission, a bonus question which I find interesting

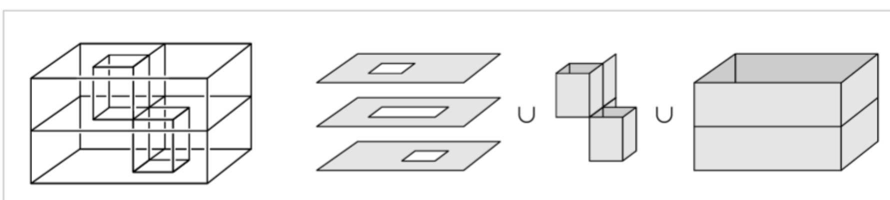
(1) **Deformation retract.** There was an inconsistency between the lecture and TA, from now we will use the definition from the lecture: given a subspace $Y \subseteq X$, a *deformation retract* is a map $r\colon X \to Y$ such that $\forall y \in Y,\ r(y) = y$ and a homotopy $h : X \times I \to X$ such that:

- $\forall x \in X,\ h(x, 0) = x$,
- $\forall x \in X,\ h(x, 1) = r(x)$, and
- $\forall y \in Y, t \in I,\ h(y, t) = y$

(a) Show that any continuous map $f\colon X \to Y$ can be decomposed into a composition $X \overset{i}{\hookrightarrow} M \overset{r}{\to} Y$ where i is a subspace inclusion and r is a deformation retract.

(b) ($\star\star$) suppose that f is a homotopy equivalence, show that M above can be constructed with a deformation retract onto X. That is, any homotopy equivalent spaces X, Y are deformation retracts of a joint space.

(c) Consider the following subspace $X \subseteq \mathbb{R}^3$, called the house with two rooms:



Describe how X has a deformation retract onto a point. There is no need to be extremely formal, you can use verbal explanations. (**Hint:** consider a small neighborhood $X \subseteq N$ of points distance ϵ from X, and do the deformation retract in two steps.)

(2) **Coproducts and Pushouts.** Recall that Top_* is the category of pointed topological spaces, with continuous maps that preserve the base-point.

(a) The coproduct of $X, Y \in \mathrm{Top}_*$ is called the *wedge sum* $X \vee Y$. Prove that Top_* has all coproducts and give an explicit description of $X \vee Y$. In particular, show that the forgetful $U : \mathrm{Top}_* \to \mathrm{Top}$ does not preserve coproducts.

(b) Consider the functor $(-)_+\colon \mathrm{Top} \to \mathrm{Top}_*$, given by $X_+ = X \sqcup \mathrm{pt}$ pointed by the added point. Prove that $(-)_+$ preserves coproducts.

(c) Show that Top_* has all pushouts, and that the forgetful $U : \mathrm{Top}_* \to \mathrm{Top}$ preserves pushouts.

(d) Give an explicit description of pushouts in Ch, and in particular prove that they all exist.

(3) **Long exact sequence of homology.**

(a) Suppose that $0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$ is a short exact sequence of chain complexes. Show that the map $d\colon H_n(C) \to H_{n-1}(A)$ defined in class fits into an exact sequence

$$\mathrm{H}_n(B_\bullet) \to \mathrm{H}_n(C_\bullet) \xrightarrow{d} \mathrm{H}_{n-1}(A_\bullet) \to \mathrm{H}_{n-1}(B_\bullet).$$

This finishes the construction of the long exact sequence in homology that you started in class.

(b) Show that the construction of d is natural, in the sense that a commuting diagram of short exact sequences of chain complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_\bullet & \longrightarrow & B_\bullet & \longrightarrow & C_\bullet & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A'_\bullet & \longrightarrow & B'_\bullet & \longrightarrow & C'_\bullet & \longrightarrow & 0 \end{array}$$

induces a commuting square

$$\begin{array}{ccc} \mathrm{H}_{n+1}(C_\bullet) & \xrightarrow{d} & \mathrm{H}_n(A_\bullet) \\ \downarrow & & \downarrow \\ \mathrm{H}_{n+1}(C'_\bullet) & \xrightarrow{d} & \mathrm{H}_n(A'_\bullet) \end{array}$$

(c) ($\star\star$) Define the correct categories and functors that formally make $d\colon H_{n+1}(C) \to H_n(A)$ into a natural transformation between functors.

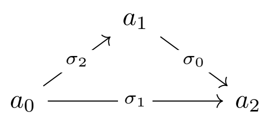
(d) (Nine Lemma) consider the following diagram of abelian groups

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A_1 & \longrightarrow & B_1 & \longrightarrow & C_1 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A_2 & \longrightarrow & B_2 & \longrightarrow & C_2 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A_3 & \longrightarrow & B_3 & \longrightarrow & C_3 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

Suppose that all the rows are exact, and the middle column is exact. Prove that the first column is exact if and only if the last column is exact.

(4) **Filling holes**.

(a) Let X be a semi-simplicial set, and let $\sigma_0, \sigma_1, \sigma_2 \in X_1$ such that $c = \sigma_0 - \sigma_1 + \sigma_2$ forms a cycle

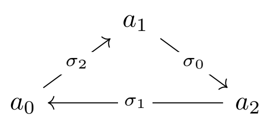


Extend X to $X' \supseteq X$ by adding a single 2-simplex $\sigma \in X'_2$ with boundaries

$$d_i(\sigma) = \sigma_i.$$

Describe $\mathrm{H}^{\Delta}_{\bullet}(X')$ in terms of $\mathrm{H}^{\Delta}_{\bullet}(X)$. (**Hint:** Split into cases based on the order of $[c] \in \mathrm{H}^{\Delta}_{1}(X)$)

(b) Suppose now that $\sigma_0, \sigma_1, \sigma_2 \in X_1$ are such that $\sigma_0 + \sigma_1 + \sigma_2$ forms a cycle



Find an extension $X'' \supseteq X$ that has the same effect on homology groups as the extension in (a) had. (**Hint:** this extension should have the same effect on geometric realization as in X'.)