Algebraic topology - Homework 3

November 20, 2024

($\star$) = not for submission, but make sure you understand how to do it
($\star\star$) = not for submission, a bonus question which I find interesting

(1) **Reduced homology** Let $\varnothing \neq X \in \mathrm{Set}_{\mathrm{s}\boldsymbol{\Delta}}$, and suppose $\mathrm{H}_0^{\Delta}(X) \simeq \mathbb{Z}^r$. The augmentation homomorphism $\epsilon\colon \mathrm{C}_0^{\Delta}(X) \to \mathbb{Z}$ was defined as $\epsilon(\sum_i n_i x_i) = \sum_i n_i$.

(a) Prove that $\widetilde{\mathrm{C}}_\bullet^{\Delta}(X)$, which is obtained by augmenting $\mathrm{C}_\bullet^{\Delta}(X)$ with ϵ,

$$\ldots \mathrm{C}_2^{\Delta}(X) \xrightarrow{\partial_2} \mathrm{C}_1^{\Delta}(X) \xrightarrow{\partial_1} \mathrm{C}_0^{\Delta}(X) \xrightarrow{\epsilon} \mathbb{Z} \to 0 \to 0 \ldots$$

is a chain complex, and show that reduced simplicial homology $\widetilde{\mathrm{H}}_n^{\Delta}(X) := \mathrm{H}_n(\widetilde{\mathrm{C}}_\bullet^{\Delta}(X))$ satisfies

$$\widetilde{\mathrm{H}}_n^{\Delta}(X) \simeq \begin{cases} \mathrm{H}_n^{\Delta}(X) & n \neq 0 \\ \mathbb{Z}^{r-1} & n = 0. \end{cases}$$

(b) ($\star\star$) Show that the isomorphism $\widetilde{\mathrm{H}}_0^{\Delta}(X) \oplus \mathbb{Z} \simeq \mathrm{H}_0^{\Delta}(X)$ cannot be realized as a natural isomorphism of functors $\mathrm{Set}_{\mathrm{s}\boldsymbol{\Delta}} \to \mathrm{Ab}$.

(c) Given $\varnothing \neq X \in \mathrm{Top}$, define *reduced singular homology* as $\widetilde{\mathrm{H}}_n^{\mathrm{Sing}}(X) := \widetilde{\mathrm{H}}_n^{\Delta}(\mathrm{Sing}(X))$. Show that reduced singular homology satisfies Mayer-Vietoris: given an open covering $X = U \cup V$ with $U \cap V \neq \varnothing$, there is a long exact sequence of reduced homology:

$$\vdots$$

$$\widetilde{\mathrm{H}}_n^{\mathrm{Sing}}(U \cap V) \longrightarrow \widetilde{\mathrm{H}}_n^{\mathrm{Sing}}(U) \oplus \widetilde{\mathrm{H}}_n^{\mathrm{Sing}}(V) \longrightarrow \widetilde{\mathrm{H}}_n^{\mathrm{Sing}}(X) \xrightarrow{d}$$

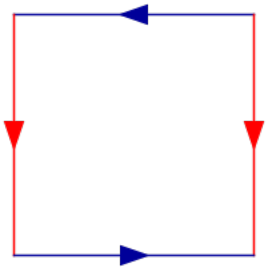
$$\widetilde{\mathrm{H}}_{n-1}^{\mathrm{Sing}}(U \cap V) \longrightarrow \widetilde{\mathrm{H}}_{n-1}^{\mathrm{Sing}}(U) \oplus \widetilde{\mathrm{H}}_{n-1}^{\mathrm{Sing}}(V) \longrightarrow \widetilde{\mathrm{H}}_{n-1}^{\mathrm{Sing}}(X) \xrightarrow{d}$$

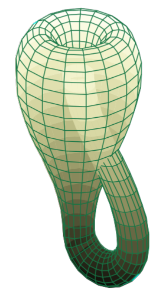
$$\vdots$$

$$\widetilde{\mathrm{H}}_0^{\mathrm{Sing}}(U \cap V) \longrightarrow \widetilde{\mathrm{H}}_0^{\mathrm{Sing}}(U) \oplus \widetilde{\mathrm{H}}_0^{\mathrm{Sing}}(V) \longrightarrow \widetilde{\mathrm{H}}_0^{\mathrm{Sing}}(X) \longrightarrow 0$$

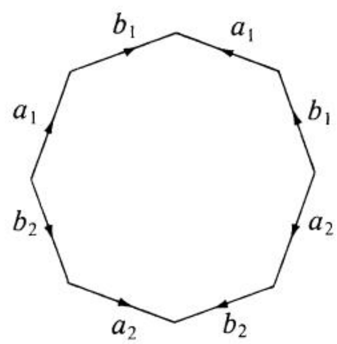
(2) **Mayer Vietoris.** In both parts, convince yourself that the construction on the left produces the space on the right.

(a) Compute the homology of the *Klein bottle*, which is given by a square with its edges glued as follows:





(b) Compute the homology of the *surface of genus* 2, which is an octagon with its edges identified as follows:



(3) **Homology of a tower.** Given an infinite sequence of Abelian groups $A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} \dots$, define its *sequential colimit* as the disjoint union of A_i where we identify every $a \in A_i$ with its image $f_i(a) \in A_{i+1}$

$$\varinjlim_i A_i := \bigsqcup_i A_i / \forall a \in A_i,\ a \sim f_i(a).$$

To define multiplication, consider $[a], [b] \in \varinjlim_i A_i$ where $a \in A_k$ and $b \in A_n$. By applying f_i enough times on a or b we may assume $n = k$, and then we define $[a] \cdot [b] = [a \cdot b]$.

(a) ($\star$) Verify that $\varinjlim_i A_i$ is a well-defined Abelian group, and show that it is the universal Abelian group with maps from the sequence

$$\begin{array}{ccccccc} A_0 & \longrightarrow & A_1 & \longrightarrow & A_2 & \longrightarrow & \cdots \\ & \searrow & & \searrow & & \searrow & \\ & & & & & & \varinjlim_i A_i. \end{array}$$

In particular, if $f_i \colon A_i \to A_{i+1}$ are inclusions of subgroups $A_i \leq A_{i+1}$, show that $\varinjlim_i A_i \simeq \bigcup_i A_i$.

(b) Given a tower of chain complexes $C^0_\bullet \leq C^1_\bullet \leq C^2_\bullet \leq \dots$, prove that

$$\mathrm{H}_n(\bigcup_i C^i_\bullet) \simeq \varinjlim_i \mathrm{H}_n(C^i_\bullet).$$

Deduce that if $X \in \mathrm{Top}$ has a tower of open subsets $U_0 \subseteq U_1 \subseteq \cdots \subseteq X$ such that $X = \bigcup_i U_i$, then

$$\mathrm{H}^{\mathrm{Sing}}_n(X) \simeq \varinjlim_i \mathrm{H}^{\mathrm{Sing}}_n(U_i).$$

Note that this sequential colimit will generally not be a union.

(c) Compute the singular homology of the infinite dimensional sphere:

$$S^\infty := \left\{ x \in \mathbb{R}^{\mathbb{N}} \;\middle|\; \begin{array}{c} x_i = 0 \text{ for all but finitely many } i \\ \sum_i x_i^2 = 1 \end{array} \right\} \subseteq \mathbb{R}^{\mathbb{N}}$$

(d) Prove that S^∞ is contractible.

(4) **Chain homotopy.**

(a) Let $f_i, g_i : A^i_\bullet \to B^i_\bullet$ be chain-homotopic maps $f_i \sim g_i$ for $i = 0, 1$. Prove that:

i. If $B^0_\bullet = A^1_\bullet$, then $f_1 \circ f_0 \sim g_1 \circ g_0$.

ii. If $A^0_\bullet = A^1_\bullet$ and $B^0_\bullet = B^1_\bullet$, then $f_0 + f_1 \sim g_0 + g_1$.

Deduce that there is a category hCh of chain complexes with equivalence classes of chain maps up to chain homotopy, and that homology factors as a functor $\mathrm{H}_n : \mathrm{hCh} \to \mathrm{Ab}$.

(b) Given $X \in \mathrm{Top}$, construct a chain homotopy between the r-th barycentric subdivision and the identity of $\mathrm{C}^{\mathrm{Sing}}_\bullet(X)$,

$$\mathrm{S}^r \sim \mathrm{id} \colon \mathrm{C}^{\mathrm{Sing}}_\bullet(X) \to \mathrm{C}^{\mathrm{Sing}}_\bullet(X).$$