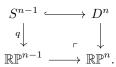
## Algebraic topology - Homework 4

November 27, 2024

 $(\star)$  = not for submission, but make sure you understand how to do it  $(\star\star)$  = not for submission, a bonus question which I find interesting

## (1) Homotopy invariance of degree.

- (a) Let  $O : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$  be an orthogonal linear transformation. In particular, O restricts to a continuous function on the unit sphere. Prove that  $\deg(O|_{S^n}) = \det(O)$ .
- (b) Suppose a continuous map  $f: S^n \to S^n$  is not surjective. Prove that  $\deg(f) = 0$ .
- (c) Suppose a continuous map  $f: S^n \to S^n$  has no fixed points. Prove that  $\deg(f) = (-1)^{n+1}$ .
- (2) **Real projective spaces.** For  $n \ge 1$ , define the *n*-dimensional real projective space as the quotient  $\mathbb{RP}^n = S^n/(-x \sim x)$ , and denote by  $q: S^n \to \mathbb{RP}^n$  the quotient map. It is enough to consider only the upper half sphere, which implies that  $\mathbb{RP}^n$  is homeomorphic to  $D^n$  where we identify antipodal points on the boundary  $S^{n-1}$ . This gives us the following inductive definition:



- (a) Calculate the singular homology of  $\mathbb{RP}^n$ , and describe the map induced by the quotient  $q_* \colon \mathrm{H}^{\mathrm{Sing}}_{\bullet}(S^n) \to \mathrm{H}^{\mathrm{Sing}}_{\bullet}(\mathbb{RP}^n).$
- (b) (\*) Deduce that there exists a homeomorphism  $\mathbb{RP}^n \simeq S^n$  only when n = 1.
- (c) Suppose  $f: S^n \to S^n$  is an even map, meaning that f(-x) = f(x) for all  $x \in S^n$ . Prove that  $\deg(f) = 0$  when n is even and  $\deg(f)$  is even when n is odd.
- (d)  $(\star\star)$  When n is odd, show that there exist an even map  $f: S^n \to S^n$  of an arbitrary even degree (start with degree 2).
- (3) **Local degree.** In this exercise, we will learn how to find the degree for a general class of maps. As a warm-up, we will start with the special case of sums of maps.
  - (a) Denote by  $E \subseteq S^n$  the equator, and note that  $S^n/E \simeq S^n \vee S^n$ . Given pointed maps  $f, g: S^n \to S^n$ , define their sum as the composition

$$f+g\colon S^n\to S^n/E\xrightarrow{\sim} S^n\vee S^n\xrightarrow{f\vee g}S^n$$

where the first map is the quotient and the third map is induced from f and g by the universal property of coproducts. Prove that  $\deg(f+g) = \deg(f) + \deg(g)$ .

(b) Let  $f: S^n \to S^n$ , and let  $x \in U \subseteq S^n$  such that  $\forall x' \in U - x$ ,  $f(x') \neq f(x)$ . Consider the following composition:

where the isomorphisms come from the exact sequence of a pair and excision. Fixing an isomorphism  $\mathrm{H}_{n}^{\mathrm{Sing}}(S^{n}) \simeq \mathbb{Z}$ , this composition becomes multiplication by some integer called the *local degree* of f at x, written  $\mathrm{deg}(f|x)$ . Now suppose  $y \in S^{n}$  has a finite pre-image  $f^{-1}(y) = \{x_{1}, \ldots, x_{k}\}$ , prove that

$$\deg(f) = \sum_{i=1}^{k} \deg(f|x_i)$$

- (c)  $(\star\star)$  Let  $p \in \mathbb{C}[x]$  be a polynomial of degree d, p induces a map on the one point compactification (Riemann sphere)  $\overline{p} \colon S^2 \to S^2$ . Show that  $\deg(\overline{p}) = d$ .
- (4) **Division algebras.** An algebra structure on  $\mathbb{R}^n$  is a bilinear multiplication map  $V \times V \to V$ . An algebra is further a *division algebra* if for every  $a, b \in \mathbb{R}^n$  with  $a \neq 0$ , the equations ax = b and xa = b have solutions.
  - (a) Show that if M is a compact n-manifold and N is a connected n-manifold, then any embedding  $M \hookrightarrow N$  is a homeomorphism.
  - (b) Suppose we have a commutative division algebra structure on  $\mathbb{R}^n$  with a multiplicative unit. Prove that  $n \leq 2$ . (**Hint:** consider the even function  $\frac{x^2}{||x^2||} \colon S^{n-1} \to S^{n-1}$ ).
  - (c) Prove that the only finite dimensional commutative division algebras over  $\mathbb{R}$  with a multiplicative unit are  $\mathbb{R}$  and  $\mathbb{C}$ .