Algebraic topology - Homework 4

November 27, 2024

($\star$) = not for submission, but make sure you understand how to do it
($\star\star$) = not for submission, a bonus question which I find interesting

(1) **Homotopy invariance of degree.**

(a) Let $O : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be an orthogonal linear transformation. In particular, O restricts to a continuous function on the unit sphere. Prove that $\deg(O|_{S^n}) = \det(O)$.

(b) Suppose a continuous map $f\colon S^n \to S^n$ is not surjective. Prove that $\deg(f) = 0$.

(c) Suppose a continuous map $f\colon S^n \to S^n$ has no fixed points. Prove that $\deg(f) = (-1)^{n+1}$.

(2) **Real projective spaces.** For $n \geq 1$, define the n-dimensional real projective space as the quotient $\mathbb{RP}^n = S^n/(-x \sim x)$, and denote by $q : S^n \to \mathbb{RP}^n$ the quotient map. It is enough to consider only the upper half sphere, which implies that $\mathbb{RP}^n$ is homeomorphic to D^n where we identify antipodal points on the boundary S^{n-1}. This gives us the following inductive definition:

$$\begin{array}{ccc} S^{n-1} & \hookrightarrow & D^n \\ {\scriptstyle q}\downarrow & & \downarrow \\ \mathbb{RP}^{n-1} & \longrightarrow & \mathbb{RP}^n. \end{array}$$

(a) Calculate the singular homology of $\mathbb{RP}^n$, and describe the map induced by the quotient $q_*\colon \mathrm{H}^{\mathrm{Sing}}_\bullet(S^n) \to \mathrm{H}^{\mathrm{Sing}}_\bullet(\mathbb{RP}^n)$.

(b) ($\star$) Deduce that there exists a homeomorphism $\mathbb{RP}^n \simeq S^n$ only when $n = 1$.

(c) Suppose $f : S^n \to S^n$ is an even map, meaning that $f(-x) = f(x)$ for all $x \in S^n$. Prove that $\deg(f) = 0$ when n is even and $\deg(f)$ is even when n is odd.

(d) ($\star\star$) When n is odd, show that there exist an even map $f : S^n \to S^n$ of an arbitrary even degree (start with degree 2).

(3) **Local degree.** In this exercise, we will learn how to find the degree for a general class of maps. As a warm-up, we will start with the special case of sums of maps.

(a) Denote by $E \subseteq S^n$ the equator, and note that $S^n/E \simeq S^n \vee S^n$. Given pointed maps $f, g\colon S^n \to S^n$, define their sum as the composition

$$f + g\colon S^n \to S^n/E \xrightarrow{\simeq} S^n \vee S^n \xrightarrow{f \vee g} S^n$$

where the first map is the quotient and the third map is induced from f and g by the universal property of coproducts. Prove that $\deg(f+g) = \deg(f) + \deg(g)$.

(b) Let $f : S^n \to S^n$, and let $x \in U \subseteq S^n$ such that $\forall x' \in U - x,\ f(x') \neq f(x)$. Consider the following composition:

$$\begin{array}{ccccc} \mathrm{H}_n^{\mathrm{Sing}}(S^n) & \xrightarrow{\sim} & \mathrm{H}_n^{\mathrm{Sing}}(S^n, S^n - x) & \xrightarrow{\sim} & \mathrm{H}_n^{\mathrm{Sing}}(U, U - x) \\ & & & & \downarrow f_* \\ & & & & \mathrm{H}_n^{\mathrm{Sing}}(S^n, S^n - f(x)) \xrightarrow{\sim} \mathrm{H}_n^{\mathrm{Sing}}(S^n) \end{array}$$

where the isomorphisms come from the exact sequence of a pair and excision. Fixing an isomorphism $\mathrm{H}_n^{\mathrm{Sing}}(S^n) \simeq \mathbb{Z}$, this composition becomes multiplication by some integer called the *local degree* of f at x, written $\deg(f|x)$. Now suppose $y \in S^n$ has a finite pre-image $f^{-1}(y) = \{x_1, \ldots, x_k\}$, prove that

$$\deg(f) = \sum_{i=1}^{k} \deg(f|x_i).$$

(c) ($\star\star$) Let $p \in \mathbb{C}[x]$ be a polynomial of degree d, p induces a map on the one point compactification (Riemann sphere) $\overline{p}\colon S^2 \to S^2$. Show that $\deg(\overline{p}) = d$.

(4) **Division algebras.** An *algebra* structure on $\mathbb{R}^n$ is a bilinear multiplication map $V \times V \to V$. An algebra is further a *division algebra* if for every $a, b \in \mathbb{R}^n$ with $a \neq 0$, the equations $ax = b$ and $xa = b$ have solutions.

(a) Show that if M is a compact n-manifold and N is a connected n-manifold, then any embedding $M \hookrightarrow N$ is a homeomorphism.

(b) Suppose we have a commutative division algebra structure on $\mathbb{R}^n$ with a multiplicative unit. Prove that $n \leq 2$. (**Hint:** consider the even function $\frac{x^2}{||x^2||} : S^{n-1} \to S^{n-1}$).

(c) Prove that the only finite dimensional commutative division algebras over $\mathbb{R}$ with a multiplicative unit are $\mathbb{R}$ and $\mathbb{C}$.