Algebraic topology - Homework 5

December 4, 2024

($\star$) = not for submission, but make sure you understand how to do it
($\star\star$) = not for submission, a bonus question which I find interesting

(1) **Operations on CW-complexes.** Let X, Y be CW-complexes, with n-cells denoted by $\{\Phi^n_\alpha \colon D^n \to X^n\}_{\alpha \in I_n}$ and $\{\Psi^n_\beta \colon D^n \to Y^n\}_{\beta \in J_n}$ respectively.

   (a) i. Define a cell structure on $X \sqcup Y$.

   ii. Suppose $Y \subseteq X$ is a subcomplex, i.e. (X, Y) is a CW-pair. Define a cell structure on X/Y.

   (b) (Leibniz rule for boundary.) Show that there is a homeomorphism $D^{n+k} \simeq D^n \times D^k$, that restricts on the boundary to a homeomorphism

$$S^{n+k-1} \simeq (S^{n-1} \times D^k) \cup_{S^{n-1} \times S^{k-1}} (D^n \times S^{k-1}).$$

   (c) Assume from now on that X and Y have only finitely many cells. Define a cell structure on $X \times Y$.

   (d) Define a cell structure on the suspension ΣX.

   (e) Suppose X, Y are pointed by 0-cells $x_0 \in X$, $y_0 \in Y$. Consider the inclusion

$$\begin{aligned} X \vee Y &\hookrightarrow X \times Y \\ x &\mapsto (x, y_0) \\ y &\mapsto (x_0, y) \end{aligned}$$

   and define the *smash product*[1] $X \wedge Y := X \times Y / X \vee Y$. Define a cell structure on $X \wedge Y$, and prove that $S^n \wedge S^k \simeq S^{n+k}$.

(2) The following parts are unrelated.

   (a) Let (X, A) be a CW-pair, and Y any space. Given $f \colon A \to Y$, denote the gluing of X to Y along f by

$$\begin{array}{ccc} A & \xrightarrow{\ f\ } & Y \\ \big\downarrow & & \big\downarrow \\ X & \longrightarrow & X \cup_{A,f} Y. \end{array}$$

[1] In latex, the symbol for the smash product is produced by `\wedge`, while the symbol for the wedge sum is produced by `\vee`.

Now suppose $f, g\colon A \to Y$ are homotopic via $h\colon A \times I \to Y$. Show that $X \cup_{A,f} Y$ and $X \cup_{A,g} Y$ are deformation retracts of $X \times I \cup_{A \times I, h} Y$.
In particular, if f and g are homotopic, then $X \cup_{A,f} Y$ and $X \cup_{A,g} Y$ are homotopy equivalent.

(b) Let $Y \subseteq X$, and suppose $X = A \cup B$, $Y = C \cup D$ are open covers such that $C \subseteq A$ and $D \subseteq B$. Construct a long exact sequence called *relative Mayer-Vietoris*

$$\cdots \to \mathrm{H}_n^{\mathrm{Sing}}(A \cap B, C \cap D) \to \mathrm{H}_n^{\mathrm{Sing}}(A, C) \oplus \mathrm{H}_n^{\mathrm{Sing}}(B, D) \to \mathrm{H}_n^{\mathrm{Sing}}(X, Y) \to \cdots.$$

Hint: Consider the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{C}_\bullet(C \cap D) & \longrightarrow & \mathrm{C}_\bullet(A \cap B) & \longrightarrow & \mathrm{C}_\bullet(A \cap B)/\mathrm{C}_\bullet(C \cap D) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{C}_\bullet(C) \oplus \mathrm{C}_\bullet(D) & \longrightarrow & \mathrm{C}_\bullet(A) \oplus \mathrm{C}_\bullet(B) & \longrightarrow & \mathrm{C}_\bullet(A)/\mathrm{C}_\bullet(C) \oplus \mathrm{C}_\bullet(B)/\mathrm{C}_\bullet(D) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{C}_\bullet{}^{C,D}(Y) & \longrightarrow & \mathrm{C}_\bullet{}^{A,B}(X) & \longrightarrow & \mathrm{C}_\bullet{}^{A,B}(X)/\mathrm{C}_\bullet{}^{C,D}(Y) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

(3) **Homology of product.** Let X be a space.

(a) ($\star$) Given a short exact sequence of abelian groups

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

prove that the following are equivalent:

i. There exists an isomorphism $B \simeq A \oplus C$, under which f becomes the inclusion of A and g becomes the projection to C.
ii. There exists $r\colon B \to A$ such that $r \circ f = \mathrm{id}_A$.
iii. There exists $s\colon C \to B$ such that $g \circ s = \mathrm{id}_C$.

A short exact sequence satisfying the equivalent conditions above is called *split*.

(b) Let X be a space and (Y, y_0) be a pointed space. Prove that

$$\mathrm{H}_i^{\mathrm{Sing}}(X \times Y) \simeq \mathrm{H}_i^{\mathrm{Sing}}(X) \oplus \mathrm{H}_i^{\mathrm{Sing}}(X \times Y, X \times \{y_0\}).$$

(c) Consider $Y = S^n$ and $y_0 = (1, 0, \ldots, 0)$, prove that

$$\mathrm{H}_i^{\mathrm{Sing}}(X \times S^n, X \times \{y_0\}) \simeq \mathrm{H}_{i-1}^{\mathrm{Sing}}(X \times S^{n-1}, X \times \{y_0\}).$$

Deduce that $\mathrm{H}_i^{\mathrm{Sing}}(X \times S^n) \simeq \mathrm{H}_i^{\mathrm{Sing}}(X) \oplus \mathrm{H}_{i-n}^{\mathrm{Sing}}(X)$ (where negative homology groups are 0).

(4) **A filtered space.** Let the spaces $B \subseteq A \subseteq X$ be as in the following drawing. Compute $\widetilde{\mathrm{H}}_\bullet(X)$, $\mathrm{H}_\bullet(X, A)$, $\mathrm{H}_\bullet(A, B)$, $\widetilde{\mathrm{H}}_\bullet(B)$.

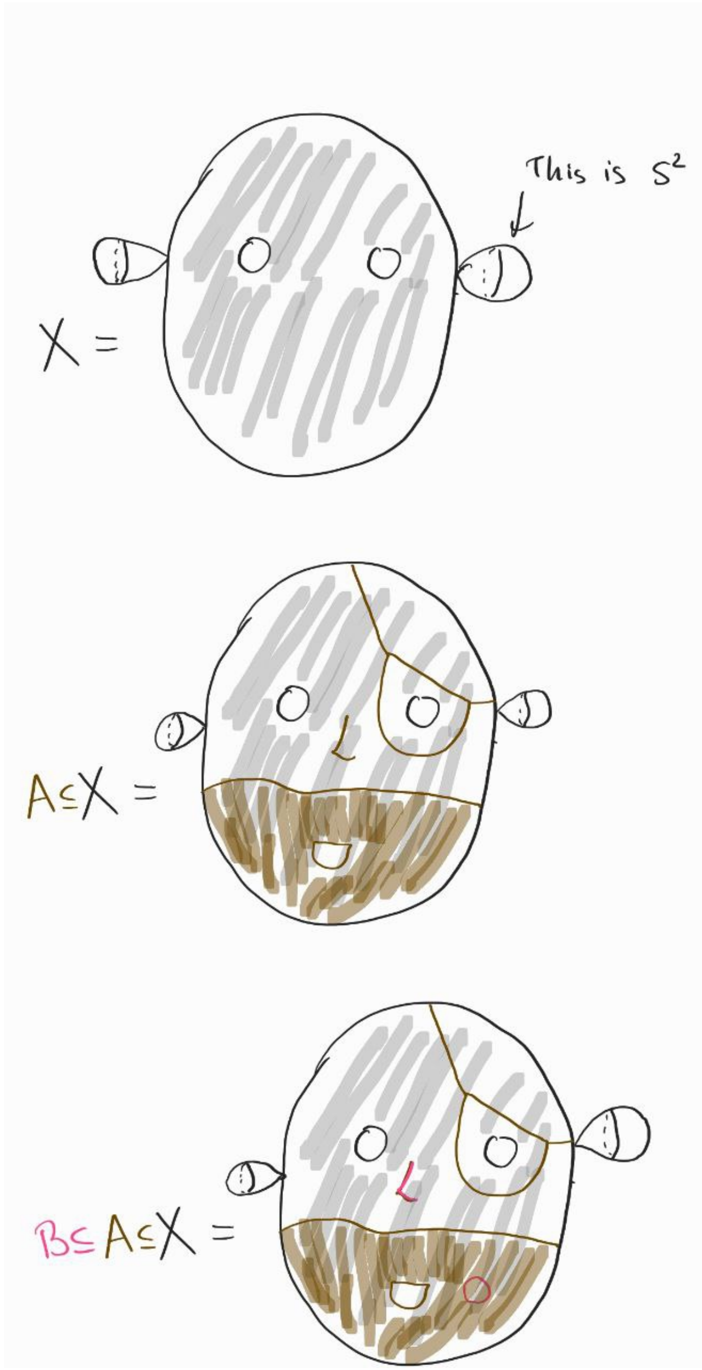

Figure 1: The Pirate