# Algebraic topology - Homework 6

December 12, 2024

($\star$) = not for submission, but make sure you understand how to do it
($\star\star$) = not for submission, a bonus question which I find interesting

(1) **Cellular homology.**

(a) Compute the homology of $\Sigma_g$, the surface of genus $g \geq 1$.

(b) Let $p$ be prime and $0 < q < p$. Define the *lens space* $L(p;q)$ as the closed 3-dimensional ball $D^3$ where we identify the northern hemisphere of the boundary with the southern hemisphere of the boundary by a reflection through the plane of the equator followed by a rotation of $\frac{2\pi q}{p}$.

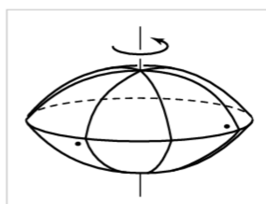

Figure 1: An illustration by Hatcher

Construct a CW structure of $L(p;q)$.

(c) Compute the homology of $L(p;q)$.

(2) **Turning the page.** Given a filtered chain complex, the $r$-page $E^r_{\bullet,\bullet}$ was defined as the $r$-cycles modulo the $r$-boundaries. Prove that the $r+1$-page is produced by taking the homology of the $r$-page.

(3) **Unbounded filtrations.** Let $C$ be a chain complex with a bi-infinite sequence of subcomplexes

$$\cdots \leq C^{(1)} \leq C^{(0)} \leq C^{(-1)} \leq \cdots \leq C.$$

We can produce an associated spectral sequence $(E^r_{n,s})$ using the same definition as in the bounded case. However, in the unbounded case we may have convergence issues.

(a) Define an induced filtration on $\bigcup_{s\in\mathbb{Z}} C^{(s)}/\bigcap_{s\in\mathbb{Z}} C^{(s)}$, and show that this filtration produces the same spectral sequence $(E^r_{n,s})$. In particular, if we want the spectral sequence to tell us something about $C$, we should demand $\bigcup_{s\in\mathbb{Z}} C^{(s)} = C$ and $\bigcap_{s\in\mathbb{Z}} C^{(s)} = 0$.

(b) Consider $C$ the chain complex

$$\cdots \to 0 \to \mathbb{Z} \xrightarrow{\times 3} \mathbb{Z} \to 0 \to \cdots$$

and consider the infinite filtration $\cdots \leq C^{(2)} \leq C^{(1)} \leq C^{(0)} = C$, where $C^{(s)}$ is the subcomplex

$$\cdots \to 0 \to 2^s\mathbb{Z} \xrightarrow{\times 3} 2^s\mathbb{Z} \to 0 \to \cdots .$$

Show that the associated spectral sequence stabilizes to 0, even though $C$ has non-zero homology.

(c) ($\star\star$) Suppose $\cdots \leq C^{(2)} \leq C^{(1)} \leq C^{(0)} = C$ is a filtered chain complex with $\bigcap_{i\in\mathbb{Z}} C^{(i)} = 0$. Consider the (categorical) limit $\hat{C}^{(s)} = \varprojlim_{i\geq s} C^{(s)}/C^{(i)}$, which is a filtration on $\hat{C} = \hat{C}^{(0)}$. Prove that $\hat{C}$ produces the same spectral sequence as $C$.

(d) Let $C$ have a filtration $0 = C^{(0)} \leq C^{(-1)} \leq C^{(-2)} \leq \cdots \leq C$ such that $\bigcup_{s\leq 0} C^{(s)} = C$. Suppose that the associated spectral sequence $(E^r_{n,s})$ stabilizes non-uniformly, meaning that for every $n, s$ there exists some $R_{n,s} \geq 0$ such that for all $r \geq R_{n,s}$ the ingoing and outgoing differentials at $E^r_{n,s}$ are 0. Define the $\infty$-page $E^\infty_{n,s} = E^{R_{n,s}}_{n,s}$ and show that $E^\infty_{n,s} = H^{(s)}_n(C)/H^{(s+1)}_n(C)$.

(e) ($\star\star$) Let $C$ have a bi-infinte filtration satisfying

$$C = \bigcup_{s\in\mathbb{Z}} C^{(s)} = \varprojlim_{s\in\mathbb{Z}} C/C^{(s)}$$

such that the associated spectral sequence stabilizes non-uniformly. Prove that $E^\infty_{n,s} = H^{(s)}_n(C)/H^{(s+1)}_n(C)$.

(4) **A filtered space II.** Recall the filtered space from last week, $B \subseteq A \subseteq X$, presented below. Compute all pages and all differentials of the associated spectral sequence.

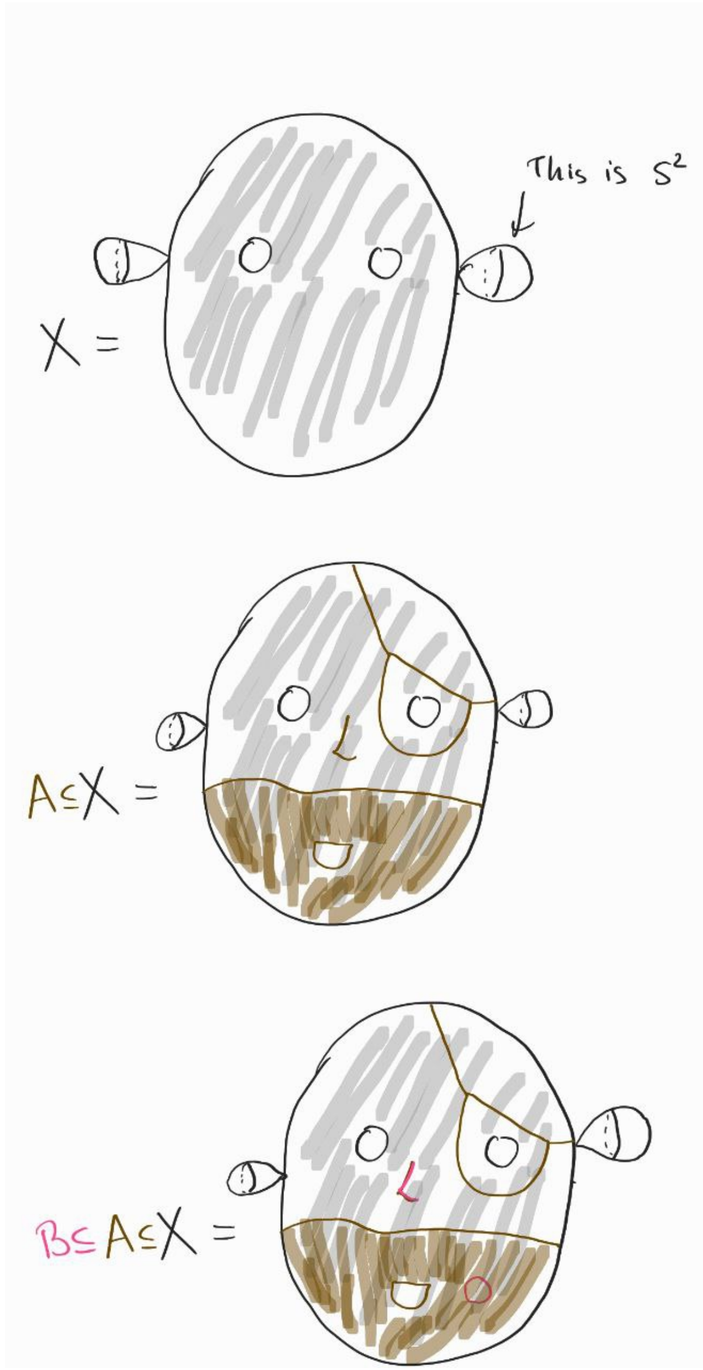


Figure 2: The Pirate