Algebraic topology - Homework 7

December 18, 2024

($\star$) = not for submission, but make sure you understand how to do it
($\star\star$) = not for submission, a bonus question which I find interesting

(1) **Bicomplexes.**

(a) The five lemma: Consider a commutative diagram of Abelian groups

$$\begin{array}{ccccccccc} A_1 & \longrightarrow & B_1 & \longrightarrow & C_1 & \longrightarrow & D_1 & \longrightarrow & E_1 \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \epsilon} \\ A_0 & \longrightarrow & B_0 & \longrightarrow & C_0 & \longrightarrow & D_0 & \longrightarrow & E_0 \end{array}$$

and assume that the rows are exact, α is surjective, ϵ is injective and β, δ are isomorphisms. Prove using spectral sequences that γ is an isomorphism.

(b) The $k \times k$ lemma: Consider a commutative diagram of Abelian groups

$$\begin{array}{ccccccccccc} & & 0 & & 0 & & & & 0 & & \\ & & \downarrow & & \downarrow & & & & \downarrow & & \\ 0 & \longleftarrow & A_{0,k-1} & \longleftarrow & A_{1,k-1} & \longleftarrow & \cdots & \longleftarrow & A_{k-1,k-1} & \longleftarrow & 0 \\ & & \downarrow & & \downarrow & & & & \downarrow & & \\ & & \vdots & & \vdots & & & & \vdots & & \\ & & \downarrow & & \downarrow & & & & \downarrow & & \\ 0 & \longleftarrow & A_{0,1} & \longleftarrow & A_{1,1} & \longleftarrow & \cdots & \longleftarrow & A_{k-1,1} & \longleftarrow & 0 \\ & & \downarrow & & \downarrow & & & & \downarrow & & \\ 0 & \longleftarrow & A_{0,0} & \longleftarrow & A_{1,0} & \longleftarrow & \cdots & \longleftarrow & A_{k-1,0} & \longleftarrow & 0 \\ & & \downarrow & & \downarrow & & & & \downarrow & & \\ & & 0 & & 0 & & & & 0 & & \end{array}$$

and assume that every row except the first $A_{\bullet,0}$, as well as every column except the first $A_{0,\bullet}$, are exact. Prove using spectral sequences that $\mathrm{H}_n(A_{\bullet,0}) \simeq \mathrm{H}_n(A_{0,\bullet})$.

(2) **Product and the fundamental group.** Let X, Y be pointed spaces, pointed by $x_0 \in X$ and $y_0 \in Y$.

(a) Prove that $\pi_1(X \times Y, (x_0, y_0)) \simeq \pi_1(X, x_0) \times \pi_1(Y, y_0)$ (**Hint:** the universal propertey of the product[1]).

(b) From the above isomorphism, it follows that a loop in $X \times \{y_0\}$ and a loop in $\{x_0\} \times Y$ represent commuting elements in $\pi_1(X \times Y, (x_0, y_0))$. Construct an explicit homotopy demonstrating this.

(3) **Conjugation and the fundamental group.** Let X be a space with $x_0 \in X$.

(a) Denote by $[S^1, X]$ the set of homotopy classes of (non-pointed) maps $S^1 \to X$, and consider the function $\pi_1(X, x_0) \to [S^1, X]$. Show that this function factors as an injection through conjugacy classes $\pi_1(X, x_0)^{\mathrm{conj}} \hookrightarrow [S^1, X]$, which is bijective if X is path connected.

(b) Let $f_t \colon X \to X$ be a homotopy between $f_0 = f_1 = \mathrm{id}_X$, and consider the loop $f_t(x_0) \colon I \to X$. Prove that $[f_t(x_0)]$ belongs to the center of $\pi_1(X, x_0)$.

(4) **Free product.**

(a) Given a set S and a subset $R \subset \langle S \rangle$, denote $\langle S \mid R \rangle = \langle S \rangle / N(R)$ where $N(R)$ is the normal closure of R. Prove that $\langle S_1 | R_1 \rangle * \langle S_2 | R_2 \rangle = \langle S_1 \sqcup S_2 | R_1 \sqcup R_2 \rangle$.

(b) Describe explicitly the pushouts in Grp, and prove that they all exist. Such pushouts are called *amalgamated* free products, and are denoted $G *_H K$.

(c) Prove that Abelaniztion $(-)^{\mathrm{ab}} \colon \mathrm{Grp} \to \mathrm{Ab}$ commutes with coproducts and pushouts.

(d) Let $F_2 = \langle \{a, b\} \rangle$, prove that the commutator subgroup $[F_2, F_2]$ is not finitely generated.

[1] is the universal property of the coproduct in the opposite category.