Algebraic topology - Homework 8

December 25, 2024

($\star$) = not for submission, but make sure you understand how to do it
($\star\star$) = not for submission, a bonus question which I find interesting

(1) **The Eckman-Hilton argument.** A unitary binary operation on a set X is a function $\otimes \colon X \times X \to X$ such that there exists a unit $e \in X$ satisfying $x \otimes e = e \otimes x = x$ for all $x \in X$.

  (a) Suppose X has two unitary binary operations $\otimes, \boxtimes \colon X \times X \to X$, which satisfy
  $$(a \otimes b) \boxtimes (c \otimes d) = (a \boxtimes c) \otimes (b \boxtimes d)$$
  for every $a, b, c, d \in X$. Prove that $\otimes$ and $\boxtimes$ are the same, and that they are in fact associative and commutative.

  (b) Let G be a topological group with unit $e \in G$, prove that $\pi_1(G, e)$ is Abelian. Deduce that $\pi_1(S^1) \simeq \mathbb{Z}$.

  (c) ($\star$) Prove that $\pi_1(G, g) \simeq \pi_1(G, e)$ for every $g \in G$.

  (d) For every $n \geq 0$, determine whether $\mathbb{R}^2$ with n points removed has a topological group structure.

(2) **Gluing cells.**

  (a) Suppose a group G has a presentation with finitely many generators and relations. Find a pointed space X such that $\pi_1(X) \simeq G$.

  (b) Let X be the space formed from the cube I^3 by identifying each face with its opposite face after applying a right-handed clockwise quarter turn. Prove that $\pi_1(X)$ is isomorphic to the quaternion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$.

(3) **Complements of knots.** Solve exercise 22 in Hatcher's Algebraic Topology, page 55.