Algebraic topology - Homework 9

January 2, 2025

($\star$) = not for submission, but make sure you understand how to do it
($\star\star$) = not for submission, a bonus question which I find interesting

(1) **Equivalence of categories.**

(a) ($\star$) A functor $F\colon \mathscr{C} \to \mathscr{D}$ is an equivalence of categories if and only if F is:

i. *Essentially surjective*: for every $Y \in \mathscr{D}$, there exists $X \in \mathscr{C}$ such that $F(X) \simeq Y$.
ii. *Faithful*: For every $X, Y \in \mathscr{C}$, $\hom_{\mathscr{C}}(X,Y) \to \hom_{\mathscr{D}}(F(X), F(Y))$ is injective.
iii. *Full*: For every $X, Y \in \mathscr{C}$, $\hom_{\mathscr{C}}(X,Y) \to \hom_{\mathscr{D}}(F(X), F(Y))$ is surjective.

The "if" direction uses the axiom of choice to construct an inverse.

(b) Let $\mathrm{Vect}_k^{\mathrm{fin}}$ be the category of finite dimensional vector spaces over a field k. Construct an equivalence of categories between $\mathrm{Vect}_k^{\mathrm{fin}}$ and $(\mathrm{Vect}_k^{\mathrm{fin}})^{\mathrm{op}}$.

(c) Recall the definition of the category $\mathrm{B}G$ for a group G from homework 1. Prove that any groupoid is equivalent to a groupoid of the form $\coprod_\alpha \mathrm{B}G_\alpha$, whose objects are the disjoint union of the objects and with no morphisms added between the components.

(d) Suppose $F\colon \mathscr{C} \to \mathscr{D}$ is an equivalence of categories, and let $D\colon I \to \mathscr{C}$ be a diagram in $\mathscr{C}$. Assume that D has a colimit $(X, f_i) \in \mathscr{C}_{D/}$, prove that $(F(X), F(f_i)) \in \mathscr{D}_{F\circ D/}$ is a colimit of $F \circ D\colon I \to \mathscr{D}$.

(e) Recall that a diagram $D\colon \mathrm{B}G \to \mathrm{Set}$ corresponds to a G-set X. Prove that the colimit of D is the G-orbits of X and the limit[1] of D is the G-fixed points of X.

(2) **Pulling back the covers.** Given continuous maps $f\colon Y \to X$, $g\colon Z \to X$ define the *pullback*

$$Y \times_X Z = \{(y,z) \in Y \times Z \mid f(y) = g(z)\}$$

with the subset topology from $Y \times Z$. We often call the projection $Y \times_X Z \to Y$ the *pullback of g along f*, in which case it is denoted f^*g.

(a) ($\star$) Prove that the pullback is the limit of the diagram given by f and g:

$$\begin{array}{ccc} Y \times_X Z & \longrightarrow & Z \\ {\scriptstyle f^*g}\downarrow & \lrcorner & \downarrow{\scriptstyle g} \\ Y & \xrightarrow[f]{} & X \end{array}$$

[1]Limits are dual to colimits, namely the limit is a terminal cone. Read any source for a more explicit description.

(b) Suppose $g\colon Z \to X$ is a covering map. Prove that the pullback $f^*g\colon Y \times_X Z \to Y$ is also covering map.

(c) Suppose X, Y are connected and locally simply connected, and let F be the $\pi_1(X)$-set associated to g. Prove that the $\pi_1(Y)$-set associated to f^*g is the same set F, with its action induced by the homomorphism $f_*\colon \pi_1(Y) \to \pi_1(X)$.

(3) **Coverings of the torus.** Given an integer matrix $A \in \mathrm{M}_{2\times 2}(\mathbb{Z})$, the induced linear map $A\colon \mathbb{R}^2 \to \mathbb{R}^2$ satisfies $A(\mathbb{Z}^2) \subseteq \mathbb{Z}^2$. It follows that A defines a continuous map on the cosets $A\colon \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2/\mathbb{Z}^2$, where $\mathbb{R}^2/\mathbb{Z}^2 \simeq \mathbb{T}$ is the torus.

(a) Suppose A is invertible as a rational matrix. Prove that the associated map $A\colon \mathbb{T} \to \mathbb{T}$ is a covering map. What is the size of the fibers?

(b) Describe the $\mathbb{Z}^2$-set associated to the covering $A\colon \mathbb{T} \to \mathbb{T}$.

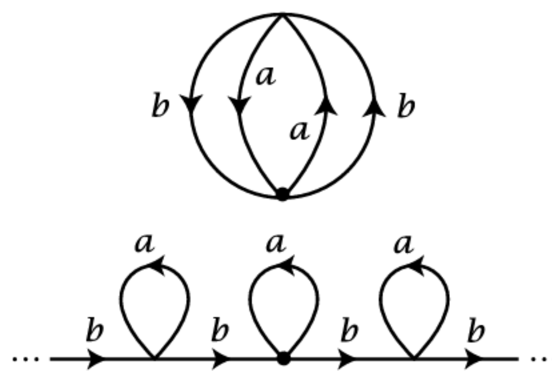
(c) Denote $\Delta\colon S^1 \to S^1 \times S^1 \simeq \mathbb{T}$ the diagonal map $\Delta(x) = (x, x)$. Consider the covering of S^1 given by the pullback $\Delta^* A$. How many orbits does the associated $\mathbb{Z}$-set has?

(4) **Coverings of $S^1 \vee S^1$.**

(a) Let Γ be a 4-regular directed graph whose edges are labeled a and b, such that every vertex has an ingoing and outgoing edge labeled a and an ingoing and outgoing edge labeled b. Show that Γ corresponds to a covering of $S^1 \vee S^1$, where all edges labeled a are mapped to one circle and all edges labeled b are mapped to the other.

(b) Find a simply connected covering space of $S^1 \vee S^1$.

(c) Consider the following (non-connected) graph:



Describe the F_2-set associated to its covering of $S^1 \vee S^1$.

(d) Find a covering space of $S^1 \vee S^1$ whose associated F_2-set is $\mathbb{Z}$ with the action $a.n = n+1$ and $b.n = n - 2$ for $n \in \mathbb{Z}$.