Algebraic topology - Homework 10

January 8, 2025

($\star$) = not for submission, but make sure you understand how to do it
($\star\star$) = not for submission, a bonus question which I find interesting

(1) **The equivalence theorem.** Let X be path connected locally simply connected space, and let $G = \pi_1(X)$.

(a) In class, you constructed an equivalence of categories $\mathrm{fib}\colon \mathrm{Cov}_X \to \mathrm{Set}_G$. Construct an explicit inverse $\mathrm{Set}_G \to \mathrm{Cov}_X$.

(b) Let $p\colon Y \to X$ and $q\colon Z \to X$ be coverings with fibers $\mathrm{fib}(Y), \mathrm{fib}(Z) \in \mathrm{Set}_G$. Prove that the pullback $Y \times_X Z \to X$ is a covering of X, and find its fiber $\mathrm{fib}(Y \times_X Z)$.

(c) A *section* of $p\colon Y \to X$ is a map $s\colon X \to Y$ such that $p \circ s = \mathrm{id}_X$. Show that there is a bijection between sections of p and fixed points of $\mathrm{fib}(Y)$.

(d) Prove that p is a homeomorphism if and only if $\mathrm{fib}(Y)$ is trivial, and Y is connected if and only if $\mathrm{fib}(Y)$ is transitive.

(2) **Klein bottle.** Let K be the Klein bottle.

(a) Prove that the simply connected cover of K is $\mathbb{R}^2$, and show that the group of deck transformation is given by the semidirect product

$$\mathrm{Aut}_K(\mathbb{R}^2) \simeq \mathbb{Z} \rtimes_\sigma \mathbb{Z}$$

where $\sigma\colon \mathbb{Z} \to \mathrm{Aut}(\mathbb{Z})$ is given by $\sigma(i) = (-1)^i$.

(b) Show that the Klein bottle is given by gluing two Möbius strips along their boundary, and prove that $\mathbb{Z} \rtimes_\sigma \mathbb{Z} \simeq \mathbb{Z} *_{2\mathbb{Z}} \mathbb{Z}$.

(c) Construct non-normal coverings of the Klein bottle by the torus and by the Klein bottle.

(3) **Free groups II.** Let $\mathbb{F}_n$ be the free group on n generators. Recall that coverings of $\bigvee_{i=1}^n S^1$ are given by graphs.

(a) Suppose $N \trianglelefteq \mathbb{F}_n$ is a non-trivial normal subgroup of infinite index, prove that N is not finitely generated. Deduce that the commutator subgroup $[\mathbb{F}_n, \mathbb{F}_n]$ is not finitely generated.

(b) For $n = 2$, draw a graph covering $S^1 \vee S^1$ whose fundamental group is $[\mathbb{F}_2, \mathbb{F}_2]$, and describe its fiber.

(c) Find explicit generators of $[\mathbb{F}_2, \mathbb{F}_2]$.

(4) **Coverings of non-connected spaces.**

(a) ($\star$) Given an equivalence of categories $F\colon \mathscr{C} \xrightarrow{\sim} \mathscr{D}$ and another category $\mathscr{E}$, prove that precomposition with F defines an equivalence of categories $\mathrm{Fun}(\mathscr{D}, \mathscr{E}) \xrightarrow{\sim} \mathrm{Fun}(\mathscr{C}, \mathscr{E})$.

(b) Let X be a locally simply connected space. Show that there is equivalence of categories between Cov_X and $\mathrm{Fun}(\pi_{\leq 1}(X)^{\mathrm{op}}, \mathrm{Set})$.

(c) Given a covering $Y \to X$, describe the corresponding functor $\pi_{\leq 1}(X)^{\mathrm{op}} \to \mathrm{Set}$ explicitly on objects and morphisms.