Algebraic topology - Homework 11

January 16, 2025

($\star$) = not for submission, but make sure you understand how to do it
($\star\star$) = not for submission, a bonus question which I find interesting

(1) **Long exact sequence in homotopy.** Given a fibration $p\colon (E, e) \to (B, b)$ with fiber $F = p^{-1}(b)$, pointed by $e \in F$, you constructed in class the boundary map $d\colon \pi_{i+1}(B) \to \pi_i(F)$.

(a) Finish the proof from class that the sequence

$$\cdots \to \pi_{n+1}(B) \xrightarrow{d} \pi_n(F) \to \pi_n(E) \to \pi_n(B) \xrightarrow{d} \pi_{n-1}(F) \to \cdots$$

is exact. Namely, prove that it is exact at d.

(b) Show that $\pi_0(F)$ has a $\pi_1(B)$-action. Moreover, show that $d\colon \pi_1(B) \to \pi_0(F)$ is given by $g \mapsto g.[e]$.

(2) **Loop space.** Given a pointed space (X, x_0), the *loop space* ΩX is the space of loops in X based on x_0. In this exercise you will show that $\pi_n(\Omega X) \simeq \pi_{n+1}(X)$.

(a) ($\star$) The *mapping space* X^Y is the set $\hom_{\mathrm{Top}}(Y, X)$ endowed with the compact-open topology. Suppose Y is locally compact, prove that there is a (natural) bijection

$$\hom_{\mathrm{Top}}(Z, X^Y) \simeq \hom_{\mathrm{Top}}(Z \times Y, X).$$

(b) The *free path space* of X is defined as X^I. Prove that the map $q\colon X^I \to X \times X$, sending a path in X to its source and target, is a fibration. The loop space is defined as the fiber $\Omega X := q^{-1}(x_0, x_0)$.

(c) Let $p\colon E \to B$ be some fibration with fiber F, and $f\colon B' \to B$ a pointed map. Prove that the pullback $f^*p\colon B' \times_B E \to B'$ is also a fibration with fiber homeomorphic to F.

(d) Consider the map $i\colon X \to X \times X$ given by $x \mapsto (x_0, x)$. The pullback

$$\begin{array}{ccc} PX & \longrightarrow & X \times I \\ {\scriptstyle i^*q}\downarrow & \lrcorner & \downarrow{\scriptstyle q} \\ X & \xrightarrow{\;i\;} & X \times X \end{array}$$

is called the *based path space*. Prove that PX is contractible. Deduce that $\pi_n(\Omega X) \simeq \pi_{n+1}(X)$.

(e) ($\star\star$) Prove that $\Omega\colon \mathrm{Top}_* \to \mathrm{Top}_*$, where ΩX is pointed by the trivial loop, has a left adjoint called the *reduced suspention* $\tilde{\Sigma} X := X \wedge S^1$.

(3) **Classifying spaces of groups.** In this exercise, you will see how group theory lives inside homotopy theory.

(a) ($\star$) For a set X, define a semisimplicial set $\vec{E}X \in \mathrm{Set}_{\mathrm{s}\boldsymbol{\Delta}}$ whose n-simplicies are $\vec{E}X_n = X^{n+1}$, and the boundary maps $d_i\colon X^{n+1} \to X^n$ removes the i-th coordinate for $0 \leq i \leq n$. Verify that $\vec{E}X$ is a semisimplicial set, and define $EX = |\vec{E}X|$ its geometric realization.

(b) Describe EX explicitly, and prove that it is contractible, assuming $X \neq \varnothing$.

(c) Given a group G, show that EG has a topologically free G-action, and define the classifying space $BG := EG/G$. Prove that

$$\pi_n(BG) = \begin{cases} G & i = 1 \\ 0 & i \neq 1 \end{cases}$$

and in particular $\pi_{\leq 1} BG$ is the groupoid we denoted $\mathrm{B}G$.

(d) ($\star\star$) Show that $G \mapsto BG$ defines a fully faithful functor $\mathrm{Grp} \to \mathrm{hTop}_*$.

(4) **Grassmannian.** Given $n \leq k$, let $V_n(\mathbb{R}^k) \subseteq (\mathbb{R}^k)^n$ be the space of orthonormal n-tuples, with the subset topology. Consider the equivalence relation $\sim$ on $V_n(\mathbb{R}^k)$, where two orthonormal n-tuples are equivalent if they span the same subspace. Define the *Grassmannian* $G_n(\mathbb{R}^k) = V_n(\mathbb{R}^k)/\sim$, which is the space of n-dimensional subspaces of $\mathbb{R}^k$.

(a) ($\star$) Notice that $G_1(\mathbb{R}^k) = \mathbb{RP}^{k-1}$, thus Grassmannians generalize projective spaces.

(b) Prove that the quotient map $V_n(\mathbb{R}^k) \to G_n(\mathbb{R}^k)$ is a fiber bundle, with fiber $O(n)$.

(c) Recall that $\mathbb{R}^\infty = \bigcup_{n<\infty} \mathbb{R}^n$ consists of infinite vectors which are zero except at finitely many places. Prove that $V_n(\mathbb{R}^\infty)$ is contractible.

(d) Deduce that $\pi_i(O(n)) = \pi_{i+1}(G_n(\mathbb{R}^\infty))$[1], and find all homotopy groups of $\mathbb{RP}^\infty$.

[1] It is possible to define classifying spaces for topological groups, after which $G_n(\mathbb{R}^\infty)$ is homotopy equivalent to $BO(n)$.