Algebraic topology - Homework 12

January 23, 2025

($\star$) = not for submission, but make sure you understand how to do it
($\star\star$) = not for submission, a bonus question which I find interesting
In this exercise there are 3 questions, each worth $33\frac{1}{3}$ points.

(1) **Basic Tor and Ext.** Let A, B, C, D be Abelian groups.

(a) Prove that Tor_1 is symmetric, namely $\mathrm{Tor}_1(A, B) \simeq \mathrm{Tor}_1(B, A)$.

(b) Let $0 \to B \to C \to D \to 0$ be a short exact sequence, prove that there are exact sequences

$$0 \to \mathrm{Tor}_1(A, B) \to \mathrm{Tor}_1(A, C) \to \mathrm{Tor}_1(A, D) \to A \otimes B \to A \otimes C \to A \otimes D \to 0,$$

$$0 \to \hom(A, B) \to \hom(A, C) \to \hom(A, D) \to \mathrm{Ext}^1(A, B) \to \mathrm{Ext}^1(A, C) \to \mathrm{Ext}^1(A, D) \to 0.$$

(c) Suppose B is *torsion free*, namely for every $0 \neq b \in B$ and $n > 0$, $nb \neq 0$. Show that $B \to B \otimes \mathbb{Q}$ is injective, and prove that $\mathrm{Tor}_1(A, \mathbb{Q}) = 0$ for every A.

(d) For an Abelian group A, define the *torsion subgroup* $A_{\mathrm{tor}} = \{a \in A \mid \exists n > 0,\ na = 0\}$. Show that A/A_{tor} is torsion free, and prove that $\mathrm{Tor}_1(A, \mathbb{Q}/\mathbb{Z}) = A_{\mathrm{tor}}$ for every A.

(e) Prove that $\mathrm{Ext}^1(\mathbb{Q}/\mathbb{Z}, \mathbb{Z}) \neq 0$.

(2) **Universal coefficients theorem.**

(a) In HW6 you defined a CW-structure on the lens space $L(p; q)$. Using this structure, calculate the homology and cohomology of $L(p; q)$ with coefficients in $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{F}_l$ for all primes l, and verify that the universal coefficients theorem holds.

(b) For a space X, define *reduced homology with coefficients* as $\widetilde{\mathrm{H}}(X; A) := \mathrm{H}_n(\widetilde{\mathrm{C}}_\bullet(X) \otimes A)$. Prove that $\widetilde{\mathrm{H}}_n(X) = 0$ for all n if and only if $\widetilde{\mathrm{H}}_n(X; \mathbb{Q}) = 0$ and $\widetilde{\mathrm{H}}_n(X; \mathbb{F}_l) = 0$ for all n and all primes l.

(3) **Tor over a ring.** For R be a commutative ring, let Mod_R be the category of R-modules and R-linear maps. For every $M, N \in \mathrm{Mod}_R$ there exists a unique (up to isomorphism) R-module $M \otimes_R N$ such that R-linear maps $M \otimes_R N \to L$ correspond to R-bilinear maps $M \times N \to R$. Denote by $\mathrm{Ch}(R)$ the category of chain complexes of R-modules, which has an induced tensor product $(C \otimes_R D)_n = \bigoplus_{i+j=n} C_i \otimes_R D_j$. An R-module is called *free* if it is of the form $\bigoplus_\alpha R$, and a *free resolution* of $M \in \mathrm{Mod}_R$ is a non-negatively graded $P \in \mathrm{Ch}_{\geq 0}(R)$ with a quasi-isomorphism $P \to \underline{M}$ such that P_i are free. Given such free resolution, define $\mathrm{Tor}_n^R(M, N) = \mathrm{H}_n(P \otimes_R \underline{N})$, which does not depend on P.

(a) ($\star$) Make sure you understand how the above claims translate from Abelian groups to general R-modules.

(b) Prove that every $M \in \mathrm{Mod}_R$ has a free resolution, and calculate $\mathrm{Tor}_n^{\mathbb{Z}/4}(\mathbb{Z}/2, \mathbb{Z}/2)$.

(c) Let $C \in \mathrm{Ch}_{\geq 0}(R)$ be a non-negatively graded chain complex of free R-modules, and let $M \in \mathrm{Mod}_R$. Construct a converging spectral sequence such that:

   i. $E^2_{n,s} \simeq \mathrm{Tor}_s^R(M, \mathrm{H}_{n-s}(C))$,

   ii. $E^\infty_{n,s}$ is the associated graded of a filtration on $H_n(M \otimes_R C)$.

(d) Let F be a field, and let $C, D, E \in \mathrm{Ch}_{\geq 0}(F)$ be non-negatively graded chain complex of F-vector spaces. Prove that:

   i. $\mathrm{H}^n(C) \simeq \mathrm{H}_n(C)^* := \hom_F(\mathrm{H}_n(C), F)$,

   ii. $\mathrm{H}_n(C \otimes_F D) \simeq \bigoplus_{i+j=n} \mathrm{H}_i(C) \otimes_F \mathrm{H}_j(D)$.

   iii. If $D \to E$ is a quasi-isomorphism, then $C \otimes_F D \to C \otimes_F E$ is a quasi-isomorphism.