Algebraic topology - Homework 13

January 30, 2025

$(\star)$ = not for submission, but make sure you understand how to do it
$(\star\star)$ = not for submission, a bonus question which I find interesting

(1) **Graded commutativity of the cup product.** A graded ring E^* is called *graded commutative* if for every $a, b \in E^*$, $ab = (-1)^{|a||b|}ba$.

(a) Define an involution $i\colon \Delta^n \to \Delta^n$ by $i(t_0, \dots, t_n) = (t_n, \dots, t_0)$. Compute $\epsilon_n := \deg(S^{n-1} \simeq \partial\Delta^n \xrightarrow{i} \partial\Delta^n \simeq S^{n-1})$.

(b) Let $X \in \mathrm{Top}$. Define an involution $\rho\colon \mathrm{C}^{\mathrm{Sing}}_\bullet(X) \to \mathrm{C}^{\mathrm{Sing}}_\bullet(X)$ by $\rho_n(\sigma) = \epsilon_n\overline{\sigma}$, where $\overline{\sigma}$ is the composition $\Delta^n \xrightarrow{i} \Delta^n \xrightarrow{\sigma} X$. Prove that ρ is a map of chain complexes.

(c) Define $P\colon \mathrm{C}^{\mathrm{Sing}}_n(X) \to \mathrm{C}^{\mathrm{Sing}}_{n+1}(X)$ by

$$P(\sigma) = \sum_{i=0}^{n} (-1)^i \epsilon_{n-i} \sigma\big|_{[0,\dots,i-1,i,n,\dots,i+1,i]},$$

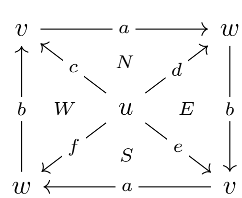
where by the restriction we mean precomposition with the map $\Delta^{n+1} \to \Delta^n$ given by $(t_0, \dots, t_{n+1}) \mapsto (t_0, \dots t_{i-1}, t_i + t_{n+1}, t_n, \dots, t_{i+1})$. Prove that P is a chain homotopy between ρ and id.

(d) Let R be a commutative ring. Show that the induced map of cochain complexes $\rho\colon \mathrm{C}^\bullet_{\mathrm{Sing}}(X;R) \to \mathrm{C}^\bullet_{\mathrm{Sing}}(X;R)$ satisfies

$$\rho(\varphi \smile \psi) = (-1)^{|\varphi||\psi|}\rho(\psi) \smile \rho(\varphi).$$

Deduce that $\mathrm{H}^*(X;R)$ is graded commutative.

(2) **Cohomology of the real projective plane.** Consider the following semisimplicial set which realizes to $\mathbb{RP}^2$:



Prove that there is an isomorphism of graded rings $\mathrm{H}^*(\mathbb{RP}^2;\mathbb{F}_2) \simeq \mathbb{F}_2[x]/(x^3)$, where x is a generator in degree 1. (**Hint:** It may help to first calculate the cohomology groups using cellular cohomology).

(3) **Classifying space of first cohomology.** Let X be a CW-complex. In this exercise, you will show that elements of $\mathrm{H}^1(X;\mathbb{Z})$ are classified by maps $X \to S^1$ up to homotopy.

(a) ($\star$) Consider S^1 as the unit circle in $\mathbb{C}$. Show that the set of homotopy classes of maps $[X, S^1]$ inherits a group structure by multiplication on the target.

(b) Show that any continuous map $f\colon X \to S^1$ is homotopic to a continuous map $\tilde{f}\colon X \to S^1$ that sends the 0-cells to the basepoint $\tilde{f}(X^0) = \{1\}$.

(c) Suppose $f\colon X \to S^1$ sends the 0-cells to the basepoint. Every 1-cell $e\colon I \to X$ defines a loop in S^1 by $f \circ e\colon I \to S^1$, where indeed $f(e(0)) = f(e(1)) = 1$. Define a cellular 1-cochain $\psi_f \in \mathrm{C}^1_{\mathrm{CW}}(X;\mathbb{Z})$ by

$$\psi_f(e) = [f \circ e] \in \pi_1(S^1, 1) \simeq \mathbb{Z}.$$

Prove that the assignment $f \mapsto \psi_f$ induces an isomorphism $[X, S^1] \xrightarrow{\sim} \mathrm{H}^1(X;\mathbb{Z})$.

(d) ($\star\star$) Let A be an Abelian group. Define a topological group structure on the classifying space BA, and prove that $[X, BA] \simeq \mathrm{H}^1(X; A)$.

(4) **Cohomological Mayer-Vietoris.** Let X be a space with an open cover $X = U \cup V$, and let A be an Abelian group. Use the universal coefficient theorem to construct a long exact sequence

$$\cdots \to \mathrm{H}^n(X;A) \to \mathrm{H}^n(U;A) \oplus \mathrm{H}^n(V;A) \to \mathrm{H}^n(U \cap V;A) \to \mathrm{H}^{n+1}(X;A) \to \cdots$$