Algebraic topology - Recitation 1

November 11, 2024

Administration

- My email address is leor.neuhauser@mail.huji.ac.il.
- Office hours Tuesday 11-12 at Ross 30.
- Homework assignments are to be typed (both Hebrew and English are okay).
- $n \in \{12, 13\}$ assignments.
- Final homework grade is sum of all assignments divided by $n - 2$, maximum 100.
- Final grade is 20% homework grade and 80% home exam.
- Assignments will be uploaded before Wednesday at 23:59 and are due by the next Wednesday at 23:59.
- Extensions are possible only by request up to 24 hours before the deadline, by email to both me and Lior Yanovski.
- Bonus questions are for general interest and will not be graded.

1 Category theory

Category theory is a framework that helps organize many different areas of mathematics, originally invented by Eilenberg and Mac Lane for algebraic topology. In this course we will use the language of category theory. We do not assume that you have taken the course on category theory, so we will give a quick overview of the relevant definitions and constructions as we will need them. We will roughly follow Lior's notes on category theory, that are available to you, and you can consult them for a more thorough exposition. Other recommended texts are

- "Handbook of categorical algebra" – Francis Borceux
- "Category theory in context" – Emily Riehl

You are also encouraged to ask questions in the office hours.

1.1 Categories

Mathematics is the art of analogy and abstraction. More precisely, you might have noticed that most undergraduate math courses begin similarly:

(1) Show examples of similar structures (e.g. $\mathbb{R}^2, \mathbb{R}^3 \dots$).

(2) Give an abstract definition unifying this structure (e.g. vector spaces over $\mathbb{R}$).

(3) Study the structure preserving maps (e.g. linear maps).

Category theory applies this procedure on itself.

Definition 1.1. A category $\mathscr{C}$ consists of:

- A set of objects $\mathrm{Ob}(\mathscr{C})$.
- For any two objects $X, Y \in \mathrm{Ob}(\mathscr{C})$, a set of morphisms $\hom_{\mathscr{C}}(X, Y)$.
- For any three objects $X, Y, Z \in \mathrm{Ob}(\mathscr{C})$, a function

$$\circ\colon \hom_{\mathscr{C}}(Y, Z) \times \hom_{\mathscr{C}}(X, Y) \to \hom_{\mathscr{C}}(X, Z).$$

- For every $X \in \mathrm{Ob}(\mathscr{C})$ an identity morphism $\mathrm{id}_X \in \hom_{\mathscr{C}}(X, X)$

Satisfying the following properties:

- (unitality) For every $f \in \hom_{\mathscr{C}}(X, Y)$,

$$f \circ \mathrm{id}_X = \mathrm{id}_Y \circ f = f$$

- (associativity) For every $f \in \hom_{\mathscr{C}}(X, Y)$, $g \in \hom_{\mathscr{C}}(b, Z)$ and $h \in \hom_{\mathscr{C}}(Z, W)$,

$$h \circ (g \circ f) = (h \circ g) \circ f$$

We will denote objects of $\mathscr{C}$ by $X, Y \in \mathscr{C}$, and morphisms $f \in \hom_{\mathscr{C}}(X, Y)$ by $f\colon X \to Y$. Given this abstract definition, we can formulate properties that are relevant for all categories. One such important definition is the isomorphism.

Definition 1.2. A morphism $f\colon X \to Y$ in a category $\mathscr{C}$ is called an *isomorphism* if there exists $g\colon Y \to Z$ such that

$$g \circ f = \mathrm{id}_X \qquad\qquad f \circ g = \mathrm{id}_Y$$

1.2 Large categories

Most familiar examples of categories are formed by sets with an extra structure. There is some set theoretic subtlety, as the objects cannot be the set of all sets. This can be solved by considering a hierarchy of sizes, and having a “large” category only consist of “small” sets, but we will not dwell on such issues. Some basic examples are:

- Set = Sets + functions
- Grp = groups + homomorphisms
- Top = topological spaces + continous maps (note that in this example the isomorphisms are homeomorphisms, and not simply bijective continuous functions)

More examples are created by restricting the objects:

- Fin = Finite sets + functions
- Ab = abelian groups + homomorphisms

Or by adding extra structure:

- Top_* = pointed topological spaces + continuous functions preserving the point

Note also that morphisms don't have to be literal functions between sets:

- hTop = topological spaces + equivalence classes of continuous functions up to homotopy
- For every category $\mathscr{C}$ one can define the opposite category $\mathscr{C}^{\mathrm{op}}$ with the same objects but whose morphisms go in the opposite direction:

  $$\hom_{\mathscr{C}^{\mathrm{op}}}(X, Y) = \hom_{\mathscr{C}}(Y, X).$$

  For example, $\mathrm{Set}^{\mathrm{op}}$ will have the sets as objects but morphisms $\hom(A, B)$ are functions $B \to A$.

1.3 Small Categories

Categories do not have to represent whole mathematical fields. There are interesting categories that only have a handful of objects and morphisms.

- Every poset $(P, \leq)$ corresponds to a category whose objects are P and which have either one or zero morphisms between any two objects, depending on their order

  $$\hom(x, y) = \begin{cases} \{*\} & x \leq y \\ \varnothing & \text{else} \end{cases}$$

  The identity $* \in \hom(x, x)$ exists by reflexivity and composition is defined uniquely by transitivity. In this category the only isomorphisms are identity maps, by anti-symmetry.
- In particular, any set A can be considered as a category with only identity morphisms. Such category is called *discrete*.
- The *semi-simplex* category $\mathrm{s}\boldsymbol{\Delta}$ has as objects $[n] = \{0 \leq 1 \leq \cdots \leq n\}$ for $n \geq 0$ and morphism $[n] \to [m]$ are injective order preserving maps

  $$\cdots \qquad [2] \;\substack{\longleftarrow\\\longleftarrow\\\longleftarrow}\; [1] \;\substack{\longleftarrow\\\longleftarrow}\; [0]$$

  Note that the maps $[n-1] \to [n]$ are the face maps $\delta_i^{[n]}$.

1.4 Functors

The basic observation of category theory, that mathematical structures are controlled by the structure preserving map, applies to categories themselves. The structure preserving maps between categories are called *functors*.

Definition 1.3. A functor $F\colon \mathscr{C} \to \mathscr{D}$ between two categories $\mathscr{C}$ and $\mathscr{D}$ consist of:

- A function $\mathrm{Ob}(\mathscr{C}) \to \mathrm{Ob}(\mathscr{D})$, denoted $X \mapsto F(X)$.
- For every $X, Y \in \mathrm{Ob}(\mathscr{C})$, a function $\hom_{\mathscr{C}}(X,Y) \to \hom_{\mathscr{D}}(F(X), F(Y))$, denoted $f \mapsto F(f)$.

Satisfying:

- $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$
- $F(g \circ f) = F(g) \circ F(f)$

Some basic examples of functors are forgetting a structure or a property:

- Top $\to$ Set forgetting the topology.
- Ab $\to$ Set forgetting the group structure.
- Ab $\to$ Grp forgetting being abelian.

Adjoint to them, there are functors that freely add a structure or property:

- $(-)^{\mathrm{disc}}\colon \mathrm{Set} \to \mathrm{Top}$ giving a set the discrete topology
- $\mathbb{Z}\langle - \rangle\colon \mathrm{Set} \to \mathrm{Ab}$ the free abelian group $\mathbb{Z}\langle X \rangle = \{\sum n_i x_i \mid n_i \in \mathbb{Z},\ x_i \in X\}$
- $(-)^{\mathrm{ab}}\colon \mathrm{Grp} \to \mathrm{Ab}$ the abelianization $G^{\mathrm{ab}} = G/[G,G]$.

A more sophisticated example is the fundamental group $\pi_1\colon \mathrm{Top}_* \to \mathrm{Grp}$ sending a pointed space (X,x) to the homotopy classes of loops $\pi_1(X,x)$. Note that a continuous pointed map $f : (X,x) \to (Y,y)$ induces a homomorphism of fundamental groups $\pi_1(X,x) \to \pi_1(Y,y)$. The fundamental group is one example of an algebraic invariant of a space, which are the subject of our course. All such invariants will be functors from a topological category to an algebraic category. They are called invariant because they send homeomorphic spaces to isomorphic groups; in fact, this follows generally from being a functor.

exercise 1.4. Let $F\colon \mathscr{C} \to \mathscr{D}$ be a functor and $f\colon X \to Y \in \mathscr{C}$ a morphism. If f is an isomorphism, then $F(f)$ is also an isomorphism.

Not every meaningful construction is functorial. For example, the center $Z\colon \mathrm{Grp} \to \mathrm{Set}$ is not a functor; the homomorphism $\mathbb{Z}/3 \to \Sigma_3$ does not restrict to a function $\mathbb{Z}/3 = Z(\mathbb{Z}/3) \to Z(\Sigma_3) = 0$. However, it is functorial with respect to isomorphism, i.e. it is an invariant, and all meaningful constructions are so.

We saw that posets correspond to certain small categories. Functors also respect this correspondence:

- Given posets P and Q, functors $P \to Q$ correspond to order preserving maps.

The expressive power of category theory really shines through when we consider small and large categories together:

- Considering the set $\mathbb{Z}$ as a discrete category, functors $\mathbb{Z} \to \mathrm{Ab}$ correspond to graded abelian groups.
- Functors $\mathrm{s}\boldsymbol{\Delta}^{\mathrm{op}} \to \mathrm{Set}$ correspond to semi-simplicial sets (in exercise).

$$\dots \qquad X_2 \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} X_1 \substack{\longrightarrow\\\longrightarrow} X_0$$

1.5 Natural transformations

Up to size issues, we can define a category whose objects are categories and whose morphisms are functors. It seems that our abstract journey has reached a satisfying end, but we are still missing a crucial piece of structure – the functors also have maps between them.

Definition 1.5. Let $\mathscr{C}, \mathscr{D}$ be categories and $F, G \colon \mathscr{C} \to \mathscr{D}$ functors. A *natural transformation* $\alpha : F \to G$ consists of morphism $\alpha_X \colon FX \to GX$ for every $X \in \mathscr{C}$, such that for every $f \colon X \to Y \in \mathscr{C}$ the following diagram commutes:

$$\begin{array}{ccc} FX & \xrightarrow{Ff} & FY \\ \downarrow{\scriptstyle \alpha_X} & & \downarrow{\scriptstyle \alpha_Y} \\ GX & \xrightarrow{Gf} & GY \end{array}$$

meaning that $\alpha_Y \circ Ff = Gf \circ \alpha_X$.

Some examples are:

- There are functors $\mathrm{GL}_n \colon \mathrm{Ring} \to \mathrm{Grp}$ sending a commutative ring R to the invertible $n \times n$ matrices with multiplication $\mathrm{GL}_n(R)$. In particular, $GL_1(R) = R^\times$. The determinant $\det : \mathrm{GL}_n(R) \to R^\times$ assembles into a natural transformation $\det \colon \mathrm{GL}_n \to \mathrm{GL}_1$:

$$\begin{array}{ccc} \mathrm{GL}_n(R) & \xrightarrow{GL_n(f)} & \mathrm{GL}_n(S) \\ \downarrow{\scriptstyle \det} & & \downarrow{\scriptstyle \det} \\ R^\times & \xrightarrow{f^\times} & S^\times \end{array}$$

  The square commutes because the determinant is given by a polynomial in the matrix coefficents

- (maybe) Let $O_n \colon \mathrm{Grp} \to \mathrm{Set}$ be the functor assigning to each group it's elements of order $\leq n$. The maps

$$\alpha_G \colon O_6(G) \to O_2(G)$$
$$g \mapsto g^3$$

  assemble into a natural transformation.

- We mentioned the functor assigning each space the discrete topology $(-)^{\text{disc}}\colon \text{Set} \to \text{Top}$. There is also a functor assigning the codiscrete, or trivial, topology $(-)^{\text{codisc}}\colon \text{Set} \to \text{Top}$. There is always a continuous map $X^{\text{disc}} \to X^{\text{codisc}}$, given by the identity map on X. This assembles into a natural transformation $(-)^{\text{disc}} \to (-)^{\text{codisc}}$

Definition 1.6. Let $\mathscr{C}, \mathscr{D}$ be categories. The functor category $\text{Fun}(\mathscr{C}, \mathscr{D})$ has functors from $\mathscr{C}$ to $\mathscr{D}$ as objects and natural transformations as morphisms.

Functors categories are a very powerful tool in creating new categories.

- $\text{GrAb} := \text{Fun}(\mathbb{Z}, \text{Ab})$ is the category of graded abelien group. Because $\mathbb{Z}$ has no morphisms, a natural transformation between $A_\bullet, B_\bullet : \mathbb{Z} \to \text{Ab}$. is simply a collection of homomorphisms $\alpha_n : A_n \to B_n$ without conditions.

- $\text{Set}_{\text{s}\boldsymbol{\Delta}} := \text{Fun}(\text{s}\boldsymbol{\Delta}^{\text{op}}, \text{Set})$ is the category of semi-simplicial sets.