Algebraic topology - Recitation 2

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1 Homotopy

A homotopy between two continuous maps is a continuous deformation that turns one into the other. Denote $I = [0,1]$.

Definition 1.1. Let $f, g\colon X \to Y$ be continuous maps between spaces X, Y. A *homotopy* $h\colon f \sim g$ is a continuous map $h : X \times I \to Y$ such that $h(-,0) = f$, $h(-,1) = g$.

Example 1.2. If $X = \mathrm{pt} = \{*\}$, then $f, g\colon \mathrm{pt} \to Y$ are determined by the points $f(*), g(*) \in Y$, and a homotopy $h\colon f \sim g$ is a path $h\colon I \to Y$ between $h(0) = f(*)$ and $h(1) = g(*)$. The general version should be thought of as a continuous family of paths: for every $x \in X$, there is a path $h(x,-)$ between $f(x)$ and $g(x)$.

Homotopy defines an equivalence relation between continuous maps, and we defined hTop as the category where morphisms are equivalence classes of continuous maps up to homotopy. All the invariants that we consider, e.g. singular homology $\mathrm{H}_n^{\mathrm{Sing}}\colon \mathrm{Top} \to \mathrm{Ab}$, send homotopical maps to the same homomorphism, so they refine to a functor $\mathrm{hTop} \to \mathrm{Grp}$. In particular, they will send isomorphisms in hTop to isomorphisms of groups. Let's describe the isomorphisms of hTop explicitly.

Definition 1.3. A continuous map $f\colon X \to Y$ is a *homotopy equivalence* if there exists $g\colon Y \to X$ and homotopies $f \circ g \sim \mathrm{id}_Y$, $g \circ f \sim \mathrm{id}_X$.

Showing that a map is a homotopy equivalence involves two different homotopies, and it can be quite hard to visualize. A simpler case, which will often suffice, is when one of the homotopies is an equality.

Definition 1.4. A continuous map $r\colon X \to Y$ is a *deformation retract* if there exists $i\colon Y \to X$ such that $r \circ i = \mathrm{id}_Y$ and there exists a homotopy $\mathrm{id}_X \sim g \circ f$.

Lemma 1.5. *Let $r \circ X \to Y$, $i \circ Y \to X$ be a retract of topological spaces, meaning $r \circ i = \mathrm{id}_Y$. Then i is a subspace inclusion, meaning that it is a homeomorphism onto its image.*

Proof. Consider $i\colon Y \to i(Y)$ and $r|_{i(Y)}\colon i(Y) \to Y$. the composition $r|_{i(Y)} \circ i\colon Y \to Y$ is the identity because $r \circ i = \mathrm{id}_Y$. On the other hand, for every $i(y) \in i(Y)$,

$$i \circ r|_{i(Y)}(i(y)) = i \circ r \circ i(y) = i \circ \mathrm{id}_Y(y) = i(y)$$

so $i \circ r|_{i(Y)} = \mathrm{id}_{i(Y)}$. ∎

In particular, in the context of a deformation retract $r\colon X \to Y$, we will usually assume WLOG that $Y \subseteq X$ and $i\colon Y \to X$ is the subspace inclusion. The condition $r \circ i = \mathrm{id}_Y$ then becomes $\forall y \in Y\ r(y) = y$.

Example 1.6. The unit disk $D^n = \{v \in \mathbb{R}^n \mid ||v|| \leq 1\}$ has a deformation retract to the point $\{0\}$. The retract $r\colon D^n \to \{0\}$ is the unique map $r(v) = 0$ (note that $r(0) = 0$). The composition $i \circ r\colon D^n \to D^n$ is the map $v \mapsto 0$, and the homotopy $\mathrm{id}_{D_2} \sim i \circ r$ is given by

$$\begin{aligned} h\colon D^n \times I &\to D^n \\ (v,t) &\mapsto (1-t)v \end{aligned}$$

Indeed, $h(v,0) = v$ and $h(v,1) = 0$.

A space X which has a deformation retract to a point $r\colon X \to \mathrm{pt}$ is called *contractible*.

Example 1.7. The unit sphere $S^n = \{v \in \mathbb{R}^n \mid ||v|| = 1\}$ is a deformation retraction of $\mathbb{R}^n \setminus 0$. The retract $r\colon \mathbb{R}^n \setminus 0 \to S^n$ is given by $v \mapsto \frac{v}{||v||}$, and the homotopy $\mathrm{id}_{\mathbb{R}^n \setminus 0} \sim i \circ r$ is given by

$$\begin{aligned} h\colon (\mathbb{R}^n \setminus 0) \times I &\to \mathbb{R}^n \setminus 0 \\ (v,t) &\mapsto (1-t)v + t\frac{v}{||v||} \end{aligned}$$

Indeed, $h(v,0) = v$ and $h(v,1) = \frac{v}{||v||}$.

Proposition 1.8. *For every space X, the projection $X \times I \to X$ is a deformation retract.*

Proof. Consider the inclusion $i\colon X \to X \times I$ given by $i(x) = (x,0)$. The homotopy $\mathrm{id}_{X\times I} \sim i \circ r$ is given by

$$\begin{aligned} h\colon (X \times I) \times I &\to X \times I \\ ((v,s),t) &\mapsto (v, s(1-t)) \end{aligned}$$

Indeed, $h((v,s),0) = (v,s)$ and $h((v,s),1) = (v,0)$. □

2 Coproducts and pushouts

2.1 Coproducts

The disjoint union of sets $X \sqcup Y$ has the following properties in the category Set:

(1) There exists morphisms $i_X\colon X \to X \sqcup Y$ and $i_Y\colon Y \to X \sqcup Y$

(2) Given any other $Z \in$ Set with morphisms $f\colon X \to Z$ and $g\colon Y \to Z$, there exists a unique $h\colon X \sqcup Y \to Z$ such that $h \circ i_X = f$ and $h \circ i_Y = g$.

Those conditions are called the *universal property of coproducts*, and they can be formulated in any category $\mathscr{C}$. This universal property characterize $X \sqcup Y$ up to a (unique) isomorphism.

Lemma 2.1. *Let $X_0, X_1 \in \mathscr{C}$ and suppose $C, C' \in \mathscr{C}$ come with maps $i_k \colon X_k \to C$ and $i'_k \colon X_k \to C'$ for $k = 0, 1$, such that both C and C' satisfy the universal property of coproducts. Then there exists a unique isomorphism $h : C \xrightarrow{\sim} C'$ such that $h \circ i_k = i'_k$ for $k = 0, 1$.*

Proof. The universal property of C ensures the existence of a unique map $h : C \to C'$ satisfying $h \circ i_k = i'_X$, we need to show that h is an isomorphism. The universal property of C' provides similarly a map $h' : C' \to C$ such that $h' \circ i'_k = i_k$, in particular $h \circ h' \colon C' \to C'$ satisfies $h \circ h' \circ i'_k = i'_k$ and $h' \circ h \colon C' \to C$ satisfies $h' \circ h \circ i_k = i_k$. However, $\mathrm{id}_{C'} \colon C' \to C'$ also satisfies $\mathrm{id}_{C'} \circ i'_k = i'_k$ and similarly $\mathrm{id}_C \colon C \to C$ satisfies $\mathrm{id}_C \circ i_k = i_k$, so by uniqueness $h \circ h' = \mathrm{id}_{C'}$ and $h' \circ h = \mathrm{id}_C$. □

Because coproducts, if they exist, are essentially unique, we will refer to them as *the* coproduct, and will usually denote them by $X \sqcup Y$.

Example 2.2. Besides disjoint union of sets, we have the following examples:

- In Top, the coproduct of X, Y is the disjoint union $X \sqcup Y$ with the induced topology.
- In $\mathrm{Set}_{s\Delta}$, the coproduct of X, Y is given by levelwise coproduct $(X \sqcup Y)_n = X_n \sqcup Y_n$ (exercise).
- In Ab, the coproduct of A, B is the direct sum $A \oplus B$.
- In Ch, the coproduct of $A_\bullet, B_\bullet$ is given by levelwise direct sum $(A_\bullet \oplus B_\bullet)_n = A_n \oplus B_n$

Even though the constructions above are different, the fact that they satisfy the same universal property allows us to compare them along functors. A naive definition is that a functor $F \colon \mathscr{C} \to \mathscr{D}$ preserves coproducts if $F(X \sqcup Y) = F(X) \sqcup F(Y)$. The problem is that coproducts are defined only up to isomorphism, so it is meaningless to ask for equality. A more sensible thing is to ask $F(X \sqcup Y) \simeq F(X) \sqcup F(Y)$, but even then we want to know *how* they are isomorphic. Luckily, we always have a canonical candidate.

Definition 2.3. Let $\mathscr{C}, \mathscr{D}$ be categories with coproducts and let $F \colon \mathscr{C} \to \mathscr{D}$ be a functor. For every $X, Y \in \mathscr{C}$, the morphisms

$$F(i_X) \colon F(X) \to F(X \sqcup Y)$$

$$F(i_Y) \colon F(Y) \to F(X \sqcup Y)$$

induce by the universal property of coproducts in $\mathscr{D}$ a morphism $F(X) \sqcup F(Y) \to F(X \sqcup Y)$ called the *assembly map*. We say that F *preserves coproducts* if the assembly map is an isomorphism for all $X, Y \in \mathscr{C}$.

Proposition 2.4. *The following functors preserve coproducts:*

(1) $\hom_{\mathrm{Top}}(Z, -) \colon \mathrm{Top} \to \mathrm{Set}$ *for* $Z \in \mathrm{Top}$ *connected non-empty.*

(2) $\mathrm{Sing} \colon \mathrm{Top} \to \mathrm{Set}_{s\Delta}$

(3) $\mathbb{Z}\langle - \rangle \colon \mathrm{Set} \to \mathrm{Ab}$

(4) $\mathrm{C}^{\Delta}_{\bullet} \colon \mathrm{Set}_{s\Delta} \to \mathrm{Ch}$

(5) $\mathrm{H}_n\colon \mathrm{Ch} \to \mathrm{Ab}$

(6) $\mathrm{H}_n^{\Delta}\colon \mathrm{Set}_{\mathrm{s}\boldsymbol{\Delta}} \to \mathrm{Ab}$

(7) $\mathrm{H}_n^{\mathrm{Sing}}\colon \mathrm{Top} \to \mathrm{Ab}$

Proof. (1) Let $X, Y \in \mathrm{Top}$, and consider the assembly map

$$\hom_{\mathrm{Top}}(Z, X) \sqcup \hom_{\mathrm{Top}}(Z, Y) \to \hom_{\mathrm{Top}}(Z, X \sqcup Y).$$

The inverse to the assembly map

$$\hom_{\mathrm{Top}}(Z, X \sqcup Y) \to \hom_{\mathrm{Top}}(Z, X) \sqcup \hom_{\mathrm{Top}}(Z, Y)$$

is built by noticing that every continuous map $f : Z \to X \sqcup Y$ factors through exactly one of the components $Z \to X$ or $Z \to Y$, as Z is connected and non-empty.

(2) The assembly map $\mathrm{Sing}(X \sqcup Y) \to \mathrm{Sing}(X) \sqcup \mathrm{Sing}(Y)$ is a natural transformation in $\mathrm{Set}_{\mathrm{s}\boldsymbol{\Delta}} = \mathrm{Fun}(\mathrm{s}\boldsymbol{\Delta}^{\mathrm{op}}, \mathrm{Set})$. Thus, as we saw in the exercise, to check that it is an isomorphism it is enough to check levelwise $\mathrm{Sing}(X \sqcup Y)_n \to \mathrm{Sing}(X)_n \sqcup \mathrm{Sing}(Y)_n$ This map is nothing but the previous assembly map

$$\hom_{\mathrm{Top}}(\Delta^n, X \sqcup Y) \to \hom_{\mathrm{Top}}(\Delta^n, X) \sqcup \hom_{\mathrm{Top}}(\Delta^n, Y)$$

which is an isomorphism.

(3) This follows because $\mathbb{Z}\langle - \rangle$ is a left adjoint. Explicitly, the assembly map $\mathbb{Z}\langle A \rangle \oplus \mathbb{Z}\langle A \rangle \to \mathbb{Z}\langle A \sqcup B \rangle$ which is given by $(\sum_i n_i a_i, \sum_j k_j b_j) \mapsto \sum_i n_i a_i + \sum_j k_j b_j$ is bijective.

(4) Similar to (2), $\mathrm{C}_\bullet^{\Delta}$ is given levelwise by $\mathbb{Z}\langle - \rangle$, it remains to show that the isomorphisms in Ch are the levelwise isomorphisms (exercise).

(5) Consider two chain complexes

$$\dots A_{n+1} \xrightarrow{\partial_{n+1}^A} A_n \xrightarrow{\partial_n^A} A_{n-1} \dots$$

$$\dots B_{n+1} \xrightarrow{\partial_{n+1}^B} B_n \xrightarrow{\partial_n^B} B_{n-1} \dots$$

with coproduct

$$\dots A_{n+1} \oplus B_{n+1} \xrightarrow{\partial_{n+1}^A \oplus \partial_{n+1}^B} A_n \oplus B_n \xrightarrow{\partial_n^A \oplus \partial_n^B} A_{n-1} \oplus B_{n-1} \dots$$

The assembly map $\mathrm{H}_n(A) \oplus \mathrm{H}_n(B) \to H_n(A \oplus B)$ is then explicitly

$$\ker(\partial_n^A)/\mathrm{Im}(\partial_{n+1}^A) \oplus \ker(\partial_n^B)/\mathrm{Im}(\partial_{n+1}^B) \to \ker(\partial_n^A) \oplus \ker(\partial_n^B)/\mathrm{Im}(\partial_{n+1}^A) \oplus \mathrm{Im}(\partial_{n+1}^B)$$

And generally, the map $A/B \oplus C/D \to A \oplus C/B \oplus D$ is an isomorphism (exercise).

(6) By composition $\mathrm{H}_\bullet^{\Delta} n = H_n \circ \mathrm{C}_\bullet^{\Delta}$

(7) By composition $\mathrm{H}_\bullet^{\mathrm{Sing}} n = \mathrm{H}_\bullet^{\Delta} n \circ \mathrm{Sing}$.

□

Remark 2.5. For a functor that does not preserve coproducts, consider the forgetful $U\colon \mathrm{Ab} \to \mathrm{Set}$.

2.2 Pushouts

A generalization of disjoint union is the gluing of two spaces along a common subspace. This operation, called the pushout, also has a universal property.

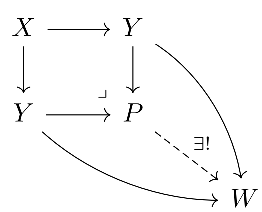
Definition 2.6. Let $X \to Y$ and $X \to Z$ be morphisms in a category $\mathscr{C}$, a *pushout square* is a commuting square

$$\begin{CD} X @>>> Y \\ @VVV @VVV \\ Z @>>> P \end{CD}$$

such that for every other commuting square

$$\begin{CD} X @>>> Y \\ @VVV @VVV \\ Y @>>> W \end{CD}$$

there exists a unique map $P \to W$ making the diagram commute



As for coproducts, the universal property ensures that the pushout is unique up to (a unique) isomorphism. Thus, we will refer to *the* pushout, and denote it by $Y \cup_X Z$

Example 2.7. Consider morphisms $f\colon X \to Y$, $g\colon X \to Z$.

(1) In Set, $Y \cup_X Z = {}^{Y \sqcup Z}\big/_{\forall x \in X, f(x) \sim g(z)}$

(2) In Top, $Y \cup_X Z = {}^{Y \sqcup Z}\big/_{\forall x \in X, f(x) \sim g(z)}$ with the quotient topology.

(3) In Ab, $B \cup_A C = {}^{B \oplus C}\big/_{\ker(f-g)}$

In general, we see that pushouts can be built from coproducts and some form of quotient. On the flip side, coproducts and quotients can be seen as forms of pushouts.

Example 2.8. The following are pushout squares in Top:

- For $X, Y \in \text{Top}$,

$$\begin{CD} \varnothing @>>> X \\ @VVV @VVV \\ Y @>>> X \sqcup Y \end{CD}$$

- For $\varnothing \neq Y \subseteq X \in \mathrm{Top}$,

$$\begin{array}{ccc} Y & \hookrightarrow & X \\ \downarrow & & \downarrow \\ \mathrm{pt} & \longrightarrow & X/Y \end{array}$$