Algebraic topology - Recitation 3

November 18, 2024

1 Simplicial Mayer-Vietoris

Today we will learn how to compute the simplicial homology of pushouts. Starting from chain complexes, in the exercise you will prove the following claim:

exercise 1.1. A commuting square of chain complexes

$$\begin{array}{ccc} A_\bullet & \xrightarrow{f} & B_\bullet \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ C_\bullet & \xrightarrow[k]{} & D_\bullet \end{array}$$

is a pushout square if and only if

$$A_\bullet \xrightarrow{f+g} B_\bullet \oplus C_\bullet \xrightarrow{h-k} D_\bullet \to 0$$

is exact. This is the same as being a pushout level-wise.

In particular, if either f or g are (level-wise) injective, then $A_\bullet \xrightarrow{f+g} B_\bullet \oplus C_\bullet$ is also injective so we get a short exact sequence of chain complexes, giving us a long exact sequence in homology. This is useful for computing simplicial homology, because both being a pushout and being injective is preserved by $\mathrm{C}_\bullet^\Delta$. We will start with the level-wise claim.

Lemma 1.2. (1) *If $f : X \to Y \in \mathrm{Set}$ is injective, then $\mathbb{Z}\langle f\rangle : \mathbb{Z}\langle X\rangle \to \mathbb{Z}\langle Y\rangle$ is injective.*

(2) *$\mathbb{Z}\langle -\rangle$ preserves pushouts.*

Proof. (1) Let $\sum_i n_i x_i \in \mathbb{Z}\langle X\rangle$ and suppose that $n_i \neq 0$ and $x_i \neq x_j$ for $i \neq j$. Then by injectivity $f(x_i) \neq f(x_j)$, and by freeness $\sum_i n_i f(x_i) \neq 0$.

(2) We will use the following fact (exercise): there is a natural isomorphism

$$\hom_{\mathrm{Ab}}(\mathbb{Z}\langle X\rangle, A) \simeq \hom_{\mathrm{Set}}(X, UA).$$

Consider a pushout of sets

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Z & \longrightarrow & W, \end{array}$$

we want to prove that

$$\begin{CD} \mathbb{Z}\langle X\rangle @>>> \mathbb{Z}\langle Y\rangle \\ @VVV @VVV \\ \mathbb{Z}\langle Z\rangle @>>> \mathbb{Z}\langle W\rangle \end{CD}$$

is a pushout square. Consider A as below,

$$\begin{tikzcd} \mathbb{Z}\langle X\rangle \arrow[r] \arrow[d] & \mathbb{Z}\langle Y\rangle \arrow[d] \arrow[ddr, bend left] & \\ \mathbb{Z}\langle Z\rangle \arrow[r] \arrow[drr, bend right] & \mathbb{Z}\langle W\rangle \arrow[dr, dashed] & \\ & & A \end{tikzcd}$$

we want to prove that the dashed arrow exists and is unique. However, mapping out of a free Abelian group is the same as mapping into the underlying set, so the data in the above diagram is equivalent to

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \arrow[ddr, bend left] & \\ Z \arrow[r] \arrow[drr, bend right] & W \arrow[dr, dashed] & \\ & & U(A) \end{tikzcd}$$

(using naturality), and this dashed arrow indeed exists uniquely.

□

Corollary 1.3. (1) *If* $f\colon X \to Y \in \mathrm{Set}_{\mathrm{s}\boldsymbol{\Delta}}$ *is (level-wise) injective, then* $\mathrm{C}^{\Delta}_{\bullet}(f)\colon \mathrm{C}^{\Delta}_{\bullet}(X) \to \mathrm{C}^{\Delta}_{\bullet}(Y)$ *is injective.*

(2) $\mathrm{C}^{\Delta}_{\bullet}$ *preserves pushouts.*

Proof. Both claims are checked level-wise, so they follow from the above Lemma. □

Theorem 1.4 (Simplicial Mayer-Vietoris)**.** Suppose

$$\begin{CD} X @>f>> Y \\ @VgVV @VVhV \\ Z @>>k> W, \end{CD}$$

is a pushout of semisimplicial sets such that f is injective. Then there exists a long exact sequence

$$\begin{array}{ccccc}
 & & \vdots & & \\
\mathrm{H}^{\Delta}_n X & \xrightarrow{f+g} & \mathrm{H}^{\Delta}_n(Y) \oplus \mathrm{H}^{\Delta}_n(Z) & \xrightarrow{h-k} & \mathrm{H}^{\Delta}_n W \\
 & & \xleftarrow{\quad d \quad} & & \\
\mathrm{H}^{\Delta}_{n-1} X & \xrightarrow[f+g]{} & \mathrm{H}^{\Delta}_{n-1}(Y) \oplus \mathrm{H}^{\Delta}_{n-1}(Z) & \xrightarrow{h-k} & \mathrm{H}^{\Delta}_{n-1} W \\
 & & \xleftarrow{\quad d \quad} & & \\
 & & \vdots & & \\
\mathrm{H}^{\Delta}_0 X & \xrightarrow{f+g} & \mathrm{H}^{\Delta}_0(Y) \oplus \mathrm{H}^{\Delta}_0(Z) & \xrightarrow{h-k} & \mathrm{H}^{\Delta}_0 W \longrightarrow 0
\end{array}$$

Proof. By the above corollary,

$$\begin{array}{ccc}
\mathrm{C}^{\Delta}_\bullet(X) & \xrightarrow{f} & \mathrm{C}^{\Delta}_\bullet(Y) \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\
\mathrm{C}^{\Delta}_\bullet(Z) & \xrightarrow[k]{} & \mathrm{C}^{\Delta}_\bullet(W),
\end{array}$$

is a pushout square, and $f\colon \mathrm{C}^{\Delta}_\bullet(X) \to \mathrm{C}^{\Delta}_\bullet(Y)$ is injective. Thus, we get a short exact sequence of chain complexes

$$0 \to \mathrm{C}^{\Delta}_\bullet(X) \to \mathrm{C}^{\Delta}_\bullet(Y) \oplus \mathrm{C}^{\Delta}_\bullet(Z) \to \mathrm{C}^{\Delta}_\bullet(W) \to 0$$

giving us the desired long exact sequence in homology. □

1.1 Union of spaces

One standard form of a pushout is taking the union over the intersection:

$$\begin{array}{ccc}
X \cap Y & \longrightarrow & X \\
\downarrow & & \downarrow \\
Y & \longrightarrow & X \cup Y
\end{array}$$

Example 1.5. Consider the *cheese burekas* $B \in \mathrm{Set}_{\mathrm{s}\mathbf{\Delta}}$

given by the pushout

$$\begin{array}{ccc} \partial\vec{\Delta}^1 & \longrightarrow & \vec{\Delta}^2 \\ \downarrow & & \downarrow \\ \vec{\Delta}^2 & \longrightarrow & B \end{array}$$

and recall that

$$\mathrm{H}_n^{\Delta}(\vec{\Delta}^2) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & \text{else} \end{cases} \qquad \mathrm{H}_n^{\Delta}(\partial\vec{\Delta}^2) = \begin{cases} \mathbb{Z} & n = 0, 1 \\ 0 & \text{else.} \end{cases}$$

By simplicial Mayer-Vietoris, we have a long exact sequence

$$\cdots 0 \to 0 \to \mathrm{H}_3^{\Delta}(B) \to 0 \to 0 \to \mathrm{H}_2^{\Delta}(B) \to \mathbb{Z} \to 0 \to \mathrm{H}_1^{\Delta}(B) \to \mathbb{Z} \to \mathbb{Z}^2 \to \mathrm{H}_0^{\Delta}(B) \to 0$$

Immediately we get $\mathrm{H}_n^{\Delta}(B) = 0$ for $n \geq 3$ and $\mathrm{H}_2^{\Delta}(B) \simeq \mathbb{Z}$. Moreover, we know $\mathrm{H}_0^{\Delta}(B) \simeq \mathbb{Z}$ as B is connected. If the map $\mathbb{Z} \to \mathbb{Z}^2$ was zero, then $\mathbb{Z}^2 \to \mathrm{H}_0^{\Delta}(B)$ would be injective, and it is also surjective so we would get an isomorphism $\mathbb{Z}^2 \simeq \mathbb{Z}$. Thus, $\mathbb{Z} \to \mathbb{Z}^2$ is not zero, so it is injective. It follows that $\mathrm{H}_1^{\Delta}(B) \to \mathbb{Z}$ is zero, so $\mathrm{H}_1^{\Delta}(B) = 0$.

We find that $\mathrm{H}_\bullet^{\Delta}(B) \simeq \mathrm{H}_\bullet^{\Delta}(\partial\vec{\Delta}^3)$, which makes sense as they both realize to S^2.

1.2 Short detour: a point in semisimplicial sets

There are two reasonable definitions for a point in a semisimplicial set. One defines it as the semisimplicial set

$$\begin{aligned} X_0 &= \mathrm{pt} \\ X_n &= \varnothing \ \forall n \neq 0. \end{aligned}$$

This deserves the name point as $|X| = \mathrm{pt}$. The other option is to take $\mathrm{Sing}(\mathrm{pt})$ which is a singleton at each level. The geometric realization of this semisimplicial set is an infinite dimensional space, so it doesn't look a lot like a point. Notice though that

$$\mathrm{C}_\bullet^{\mathrm{Sing}}(\mathrm{pt}) = \mathrm{C}_\bullet^{\Delta}(\mathrm{Sing}(\mathrm{pt})) = \cdots \xrightarrow{\mathrm{id}} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\mathrm{id}} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

thus its homology is

$$\mathrm{H}_n^{\mathrm{Sing}}(\mathrm{pt}) = \mathrm{H}_n^{\Delta}(\mathrm{Sing}(\mathrm{pt})) = \begin{cases} \mathbb{Z}, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

so it has the homology of a point. Later in the course we will show that $\mathrm{Sing}(\mathrm{pt})$ is contractible, that is homotopy equivalent to a point, so it also deserves to be called a point.

We will usually use this definition as it has a good universal property: for every $X \in \mathrm{Set}_{\mathrm{s}\boldsymbol{\Delta}}$ there is a unique map $X \to \mathrm{Sing}(\mathrm{pt})$ (because there is a unique map $|X| \to \mathrm{pt}$).

1.3 Quotient spaces

Suppose $X \subseteq Y$ is a sub-semisimplicial set, and Y/X is the level-wise quotient. The quotient fits into a pushout square

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & \lrcorner & \downarrow \\ \mathrm{pt} & \longrightarrow & Y/X. \end{array}$$

Example 1.6. Consider the qotient of the 2-simplex by it's boundary

$$\begin{array}{ccc} \partial\vec{\Delta}^2 & \longrightarrow & \vec{\Delta}^2 \\ \downarrow & \ulcorner & \downarrow \\ \mathrm{pt} & \longrightarrow & \vec{\Delta}^2/\partial\vec{\Delta}^2 = Q. \end{array}$$

Meyer-Vietoris gives us a long exact sequence

$$\cdots 0 \to 0 \to \mathrm{H}_3^\Delta(B) \to 0 \to 0 \to \mathrm{H}_2^\Delta(Q) \to \mathbb{Z} \to 0 \to \mathrm{H}_1^\Delta(Q) \to \mathbb{Z} \to \mathbb{Z}^2 \to \mathrm{H}_0^\Delta(Q) \to 0$$

Which is the same long exact sequence as for B, so by the same arguments $\mathrm{H}_\bullet^\Delta(Q) \simeq \mathrm{H}_\bullet^\Delta(B)$. This makes sense, because both Q and B realize to S^2.

1.4 The real projective plane

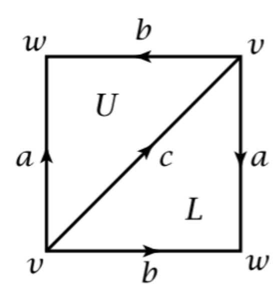
The example of real projective space will use a more complicated pushout, which isn't a union or quotient. There are multiple homeomorphic definitions of $\mathbb{RP}^2$:

(1) The space of lines in $\mathbb{R}^3$

(2) Noticing that each line intersects S^2 in two antipodal points, this is equivalent to $S^2/-x \sim x$.

(3) Each point in the lower half sphere is equivalent to a point in the upper half sphere, so we can remember only the upper half sphere, which is homeomorphic to the disk D^2. Note that on the equator we still need to identify antipodal points, so we get $\mathbb{RP}^2 \simeq D^2/(\forall x \in S^1,\ -x \sim x)$

There is a semisimplicial set X realizing to $\mathbb{RP}^2$, given by



In the exercise you were asked to calculate the simplicial homology of X directly, now we will do so with Mayer-Vietoris. First, for some intuition, notice that $a-b$ is a cycle that is not a boundary, however $2a-2b$ is the boundary of $L-U$. Thus, we excpet to find an element $[a-b] \in \mathrm{H}_1^{\Delta}(X)$ of order 2. This is a good example of why we need homology *groups* to describe the holes in a shape, instead of just counting them.

Consider the following semisimplicial sets:

$$Y = \begin{array}{ccc} w & \xleftarrow{\;b\;} & v' \\ {\scriptstyle a}\big\uparrow & {\scriptstyle U}\nearrow{\scriptstyle L} & \big\downarrow{\scriptstyle a'} \\ v & \xrightarrow[\;b'\;]{} & w' \end{array} \qquad \partial Y = \begin{array}{ccc} w & \xleftarrow{\;b\;} & v' \\ {\scriptstyle a}\big\uparrow & & \big\uparrow{\scriptstyle a'} \\ v & \xrightarrow[\;b'\;]{} & w' \end{array} \qquad \partial X = \begin{array}{c} w \\ {\scriptstyle a}\big\uparrow\;\big\uparrow{\scriptstyle b} \\ v \end{array}$$

Their homology can be calculated directly, or using Mayer-Vietoris (they are gluings of $\vec{\Delta}^2$ and $\vec{\Delta}^1$), to be

$$\mathrm{H}_n^{\Delta}(Y) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n > 0 \end{cases} \qquad \mathrm{H}_n^{\Delta}(\partial Y) \simeq \mathrm{H}_n^{\Delta}(\partial X) \simeq \begin{cases} \mathbb{Z} & n = 0, 1 \\ 0 & n > 1. \end{cases}$$

And there is a pushout square

$$\begin{array}{ccc} \partial Y & \longrightarrow & Y \\ \big\downarrow & & \big\downarrow \\ \partial X & \longrightarrow & X. \end{array}$$

By Mayer-Vietoris, we get a long exact sequence

$$\begin{aligned} &\cdots 0 \to 0 \to \mathrm{H}_3^{\Delta}(X) \to \\ &0 \to 0 \to \mathrm{H}_2^{\Delta}(X) \to \\ &\mathrm{H}_1^{\Delta}(\partial Y) \to \mathrm{H}_1^{\Delta}(\partial X) \to \mathrm{H}_1^{\Delta}(X) \to \\ &\mathrm{H}_0^{\Delta}(\partial Y) \to \mathrm{H}_0^{\Delta}(Y) \oplus \mathrm{H}_0^{\Delta}(\partial X) \to \mathrm{H}_0^{\Delta}(X) \to 0 \end{aligned}$$

Immediately we get $\mathrm{H}_n^{\Delta}(X) = 0$ for $n \geq 3$. We also know that X is connected, so $\mathrm{H}_0^{\Delta}(X) \simeq \mathbb{Z}$, and the last row is

$$\mathbb{Z} \to \mathbb{Z}^2 \to \mathbb{Z} \to 0$$

The first map can't be zero, because then the second will be injective, but it is also surjective, so it will be an isomorphism $\mathbb{Z}^2 \xrightarrow{\sim} \mathbb{Z}$. Thus, the first map must be injective. It follows that the map $\mathrm{H}_1^{\Delta}(X) \to \mathrm{H}_0^{\Delta}(\partial Y)$ is zero, giving us an exact sequence

$$0 \to \mathrm{H}_2^{\Delta}(X) \to \mathrm{H}_1^{\Delta}(\partial Y) \to \mathrm{H}_1^{\Delta}(\partial X) \to \mathrm{H}_1^{\Delta}(X) \to 0$$

We have $\mathrm{H}_1^{\Delta}(\partial Y) = \mathbb{Z}\langle [a-b+a'-b'] \rangle$ and $\mathrm{H}_1^{\Delta}(\partial X) = \mathbb{Z}\langle [a-b] \rangle$ and the map between them is $\mathbb{Z}\langle [a-b+a'-b'] \rangle \xrightarrow{[a-b+a'-b'] \mapsto 2[a-b]} \mathbb{Z}\langle [a-b] \rangle$, so we get

$$0 \to \mathrm{H}_2^{\Delta}(X) \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathrm{H}_1^{\Delta}(X) \to 0$$

Thus $\mathrm{H}_2^{\Delta}(X) \simeq \ker(\times 2) = 0$, and $\mathrm{H}_1^{\Delta}(X) \simeq \operatorname{coker}(\times 2) = \mathbb{Z}/2\mathbb{Z}$. In conclusion,

$$\mathrm{H}_n^{\Delta}(X) \simeq \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}/2\mathbb{Z} & n = 1 \\ 0 & n \geq 2. \end{cases}$$

1.5 Reduced homology

Notice that in all of the above examples, the long exact sequence ended with

$$\mathrm{H}_1^{\Delta}(X) \to \mathbb{Z} \to \mathbb{Z}^2 \to \mathrm{H}_0^{\Delta}(X) \to 0$$

and the same arguments implied that $\mathrm{H}_0^{\Delta}(X) = \mathbb{Z}$ and the map $\mathrm{H}_1^{\Delta}(X) \to \mathbb{Z}$ is zero. This is because all the relevant spaces were connected, so the 0-th homology was $\mathbb{Z}$. To make things simpler, we would like a modification of homology such that the 0-th homology off a connected space will be 0.

Definition 1.7. Given $\varnothing \neq X \in \mathrm{Set}_{\mathrm{s}\boldsymbol{\Delta}}$, the *augmentation* map $\epsilon\colon \mathrm{C}_0^{\Delta}(X) \to \mathbb{Z}$ is given by $\epsilon(\sum_i n_i x_i) = \sum_i n_i$. The *augmented simplicial chain complex* $\widetilde{\mathrm{C}}_\bullet^{\Delta}(X)$ is given by augmenting $\mathrm{C}_\bullet^{\Delta}(X)$ with ϵ in degree -1:

$$\cdots \xrightarrow{\partial_2} \mathrm{C}_1^{\Delta}(X) \xrightarrow{\partial_1} \mathrm{C}_0^{\Delta}(X) \xrightarrow{\varepsilon} \mathbb{Z} \to 0 \to 0 \cdots .$$

And *reduced homology* is defined by

$$\widetilde{\mathrm{H}}_n^{\Delta}(X) = \mathrm{H}_n(\widetilde{\mathrm{C}}_\bullet^{\Delta}(X))$$

$$\widetilde{\mathrm{H}}_n^{\mathrm{Sing}}(X) = \widetilde{\mathrm{H}}_n^{\Delta}(\mathrm{Sing}(X))$$

Recall that for $X \neq \varnothing$ $\mathrm{H}_0^{\Delta}(X) \simeq \mathbb{Z}^r$ is a free Abelian group of some rank $r > 0$. The reduced homology reduces this rank by 1.

Proposition 1.8. *Let $X \in \mathrm{Set}_{\mathrm{s}\boldsymbol{\Delta}}$, then*

$$\widetilde{\mathrm{H}}_n^{\Delta}(X) = \begin{cases} \mathrm{H}_n^{\Delta}(X), & n \neq 0 \\ \mathrm{H}_0^{\Delta}(X)/\mathbb{Z} \simeq \mathbb{Z}^{r-1}, & n = 0 \end{cases}$$

In the exercise you will see that reduced homology also satisfies simplicial Mayer-Vietoris.

Theorem 1.9 (Simplicial Mayer-Vietoris)**.** Suppose

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ Z & \xrightarrow[k]{} & W, \end{array}$$

is a pushout of non-empty semisimplicial sets such that f is injective. Then there exists a long exact sequence

$$
\begin{array}{ccccccc}
 & & \vdots & & & & \\
\widetilde{\mathrm{H}}^{\Delta}_n X & \xrightarrow{f+g} & \widetilde{\mathrm{H}}^{\Delta}_n(Y) \oplus \widetilde{\mathrm{H}}^{\Delta}_n(Z) & \xrightarrow{h-k} & \widetilde{\mathrm{H}}^{\Delta}_n W & & \\
 & & \xrightarrow{d} & & & & \\
\widetilde{\mathrm{H}}^{\Delta}_{n-1} X & \xrightarrow[f+g]{} & \widetilde{\mathrm{H}}^{\Delta}_{n-1}(Y) \oplus \widetilde{\mathrm{H}}^{\Delta}_{n-1}(Z) & \xrightarrow{h-k} & \widetilde{\mathrm{H}}^{\Delta}_{n-1} W & & \\
 & & \xrightarrow{d} & & & & \\
 & & \vdots & & & & \\
\widetilde{\mathrm{H}}^{\Delta}_0 X & \xrightarrow{f+g} & \widetilde{\mathrm{H}}^{\Delta}_0(Y) \oplus \widetilde{\mathrm{H}}^{\Delta}_0(Z) & \xrightarrow{h-k} & \widetilde{\mathrm{H}}^{\Delta}_0 W & \longrightarrow & 0
\end{array}
$$