Algebraic topology - Recitation 4

November 25, 2024

1 Naturality of Mayer-Vietoris

In the exercise, you showed that the long exact sequence in homology is natural. We will present a solution, and use it show naturality of Mayer-Vietoris, which we will need today.

Lemma 1.1. *A commuting diagram of short exact sequences of chain complexes*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_\bullet & \xrightarrow{i} & B_\bullet & \xrightarrow{j} & C_\bullet & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow g & & \downarrow h & & \\ 0 & \longrightarrow & A'_\bullet & \xrightarrow{i'} & B'_\bullet & \xrightarrow{j'} & C'_\bullet & \longrightarrow & 0 \end{array}$$

induces a commuting square

$$\begin{array}{ccc} \mathrm{H}_{n+1}(C_\bullet) & \xrightarrow{d} & \mathrm{H}_n(A_\bullet) \\ \downarrow h & & \downarrow f \\ \mathrm{H}_{n+1}(C'_\bullet) & \xrightarrow{d} & \mathrm{H}_n(A'_\bullet) \end{array}$$

Proof. Let $[c] \in \mathrm{H}_{n+1}(C_\bullet)$, we first recall how $d([c])$ is defined. By surjectivity there is some $b \in B_{n+1}$ such that $j(b) = c$, and c is a cycle, so

$$0 = \partial c = \partial j(b) = j(\partial b)$$

and so by exactness there is a (unique) $a \in A_n$ such that $i(a) = \partial b$. We defined $d([c]) = [a]$ (you showed that it is well-defined).

On the other hand, consider $h(c) \in C'_{n+1}$. We have $j'(g(b)) = h(j(b)) = h(c)$, and $f(a) \in A'_n$ satisfies $i'(f(a)) = g(i(a)) = g(\partial b) = \partial g(b)$, so $d([h(c)]) = [f(a)]$. It follows that

$$d \circ h([c])) = f([a]) = f \circ d([c])$$

□

This shows naturality at d, it is easier for the other maps in the sequence (but we also only need it at d today).

Proposition 1.2. *Suppose $X = U \cup V$, $X' = U' \cup V'$ are open coverings, and suppose $f\colon X \to X'$ restrict to $f(U) \subseteq U'$, $f(V) \subseteq V'$. Then Mayer-Vietoris is natural with respect to f:*

$$\begin{array}{ccccccc} \cdots \longrightarrow & \mathrm{H}^{\mathrm{Sing}}_{n+1}(X) & \xrightarrow{d} & \mathrm{H}^{\mathrm{Sing}}_{n}(U \cap V) & \longrightarrow \cdots \\ & \downarrow f_* & & \downarrow f_* & \\ \cdots \longrightarrow & \mathrm{H}^{\mathrm{Sing}}_{n+1}(X') & \xrightarrow{d} & \mathrm{H}^{\mathrm{Sing}}_{n}(U' \cap V') & \longrightarrow \cdots \end{array}$$

Proof. It is enough to see that we have a commuting diagram of short exact sequences of chain complexes

$$\begin{array}{ccccccccc} 0 \longrightarrow & \mathrm{C}^{\mathrm{Sing}}_\bullet(U \cap V) & \longrightarrow & \mathrm{C}^{\mathrm{Sing}}_\bullet(U) \oplus \mathrm{C}^{\mathrm{Sing}}_\bullet(V) & \longrightarrow & \mathrm{C}^{\mathrm{Sing}}_\bullet(U + V) & \longrightarrow 0 \\ & \downarrow f_* & & \downarrow f_* \oplus f_* & & \downarrow f_* & \\ 0 \longrightarrow & \mathrm{C}^{\mathrm{Sing}}_\bullet(U' \cap V') & \longrightarrow & \mathrm{C}^{\mathrm{Sing}}_\bullet(U') \oplus \mathrm{C}^{\mathrm{Sing}}_\bullet(V') & \longrightarrow & \mathrm{C}^{\mathrm{Sing}}_\bullet(U' + V') & \longrightarrow 0 \end{array}$$

□

2 Degree

Consider a continuous map $f : S^n \to S^n$ for $n \geq 0$ (where $S^0 = \{1, -1\}$). By functoriality, this induces a map on (reduced) singular homology $f_*\colon \widetilde{\mathrm{H}}^{\mathrm{Sing}}_n(S^n) \to \widetilde{\mathrm{H}}^{\mathrm{Sing}}_n(S^n)$. Fix an isomorphism $\widetilde{\mathrm{H}}^{\mathrm{Sing}}_n(S^n) \simeq \mathbb{Z}$. Under such an isomorphism, we have $f_*\colon \mathbb{Z} \to \mathbb{Z}$.

Definition 2.1. The degree of $f\colon S^n \to S^n$ is defined as $\deg(f) = f_*(1) \in \mathbb{Z}$.

A choice of a different isomorphism $\widetilde{\mathrm{H}}^{\mathrm{Sing}}_n(S^n) \simeq \mathbb{Z}$ amounts to replacing 1 with -1, but because we do it on both sides we will get the same degree. For $n > 0$, we could use unreduced homology, as $\mathrm{H}^{\mathrm{Sing}}_n(S^n) \simeq \widetilde{\mathrm{H}}^{\mathrm{Sing}}_n(S^n)$. Note moreover that for $n = 0$ there are only four maps $S^0 \to S^0$, and we can find their degrees explicitly (try!).

Proposition 2.2. *The following are basic properties of the degree:*

(1) $\deg(\mathrm{id}_{S^n}) = 1$

(2) $\deg(f \circ g) = \deg(f)\deg(g)$

(3) *The degree is homotopy invariant: If $f \sim g$ then* $\deg(f) = \deg(g)$.

(4) *If f is a homotopy equivalence, then* $\deg(f) = \pm 1$.

Proof. (1) and (2) follow from functoriality. For (2), notice that $(f \circ g)_*(1) = f_*(g_*(1)) = f_*(1)g_*(1)$. (3) follows from homotopy invariance of singular homology, and (4) from that fact that if g is homotopy inverse to f then $\deg(f)\deg(g) = 1$. □

As examples of a map with degree -1, we have reflection along a single coordinate. We will first do the case S^0.

Lemma 2.3. *Let $r\colon S^0 \to S^0$ be the reflection $1, -1 \mapsto -1, 1$. Then* $\deg(r) = -1$.

Proof. $\mathrm{Sing}(S^0)$ has only two 0-simplices, corresponding to ± 1, and the 1-simplices are constant at 1 or -1 so they have a trivial boundary. Thus, $\widetilde{\mathrm{H}}_0^{\mathrm{Sing}}(S^0)$ is the kernel of the augmentation map

$$\epsilon\colon \mathbb{Z}\langle [1], [-1] \rangle \to \mathbb{Z},$$

and in particular $\widetilde{\mathrm{H}}_0^{\mathrm{Sing}}(S^0)$ is generated by $[1] - [-1]$. The map r_* then induces $r_*([1] - [-1]) = [-1] - [1]$, so r_* acts by multiplication with -1. ☐

For the reflection in S^n, we could again compute on a generator directly, but this becomes harder. Instead, we will prove by induction using the naturality of Mayer-Vietoris.

Proposition 2.4. *Let $r : S^n \to S^n$ be given by $r(x_0, x_1, \ldots, x_n) = (-x_0, x_1, \ldots, x_n)$, then* $\deg(r) = -1$

Proof. By induction on n. The case $n = 0$ is the above Lemma, assume $n > 0$. Consider the covering of S^n given by the north and south poles

$$N = \{(x_0, \ldots, x_n) \in S^n | x_n \geq 0\} \qquad S = \{(x_0, \ldots, x_n) \in S^n | x_n \leq 0\}$$

with intersection the equator

$$E = \{(x_0, \ldots, x_n) \in S^n | x_n = 0\}.$$

Even though N and S are closed, they have neighborhoods $\tilde{N}, \tilde{S}$ that deformation retracts to them, and $\tilde{N} \cap \tilde{S}$ deformation retracts to E, so we can use them in (reduced) Mayer-Vietoris. Moreover, notice that $r(N) = N$ and $r(S) = S$, so we can use naturality.

$$\begin{array}{ccccccc}
0 = \widetilde{\mathrm{H}}_n^{\mathrm{Sing}}(N) \oplus \widetilde{\mathrm{H}}_n^{\mathrm{Sing}}(S) & \longrightarrow & \widetilde{\mathrm{H}}_n^{\mathrm{Sing}}(S^n) & \xrightarrow{\sim} & \widetilde{\mathrm{H}}_{n-1}^{\mathrm{Sing}}(E) & \longrightarrow & \widetilde{\mathrm{H}}_{n-1}^{\mathrm{Sing}}(N) \oplus \widetilde{\mathrm{H}}_{n-1}^{\mathrm{Sing}}(S) = 0 \\
 & & \downarrow{\scriptstyle r_*} & & \downarrow{\scriptstyle r_*} & & \\
0 = \widetilde{\mathrm{H}}_n^{\mathrm{Sing}}(N) \oplus \widetilde{\mathrm{H}}_n^{\mathrm{Sing}}(S) & \longrightarrow & \widetilde{\mathrm{H}}_n^{\mathrm{Sing}}(S^n) & \xrightarrow{\sim} & \widetilde{\mathrm{H}}_{n-1}^{\mathrm{Sing}}(E) & \longrightarrow & \widetilde{\mathrm{H}}_{n-1}^{\mathrm{Sing}}(N) \oplus \widetilde{\mathrm{H}}_{n-1}^{\mathrm{Sing}}(S) = 0
\end{array}$$

Notice that $E \simeq S^{n-1}$, and $r|_E$ is the reflection along the first coordinate in S^{n-1}. By the induction hypothesis $r_*\colon \widetilde{\mathrm{H}}_{n-1}^{\mathrm{Sing}}(E) \to \widetilde{\mathrm{H}}_{n-1}^{\mathrm{Sing}}(E)$ is given by multiplication by -1, so by the above isomorphism $r_*\colon \widetilde{\mathrm{H}}_n^{\mathrm{Sing}}(S^n) \to \widetilde{\mathrm{H}}_n^{\mathrm{Sing}}(S^n)$ also multiplies by -1. Thus $\deg(r) = -1$. ☐

Corollary 2.5. *The map $-\mathrm{id}_{S^n}\colon S^n \to S^n$ sending $(x_0, \ldots, x_n) \mapsto (-x_0, \ldots, -x_n)$ has degree* $(-1)^{n+1}$

Proof. It is the composition of $n + 1$ reflections. ☐

Already from this calculation we can deduce interesting results. A *continuous tangent vector field* on S^n is a continuous map $v\colon S^n \to \mathbb{R}^{n+1}$ such that for every $x \in S^n$, $x \perp v(x)$.

Theorem 2.6 (Hairy Ball Theorem)**.** There exists a non-vanishing continuous vector field on S^n (meaning $v(x) \neq 0$ for all $x \in S^1$) if and only if n is odd.

In the case $n = 2$, you can imagine that the vector field represents hairs on a hairy ball. The non-existence for $n = 2$ tells us that we cannot comb a hairy ball without having hairs pointing up (a schwanz).

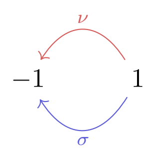
Proof. Assume that such vector field exists. Because $v(x) \neq 0$, we may normalize so that $||v(x)|| = 1$ by replacing $v(x)$ with $v(x)/||v(x)||$. We will define a homotopy id $\sim -$id by drawing the arc from x to $-x$ through $v(x)$. Explicitly, define $h : S^1 \times I \to S^1$ by $h(x,t) = \cos(\pi t)x + \sin(\pi t)v(x)$, where $h(x,t) \in S^n$ because $||h(x,t)|| = cos(\pi t)^2 + \sin(\pi t)^2 = 1$, and indeed $h(x,0) = x$ and $h(x,1) = -x$. Thus, $1 = \deg(\mathrm{id}) = \deg(-\mathrm{id}) = (-1)^n + 1$, which implies that n is odd.

For n odd, there is a non-vanishing vector field given by $v(x) = (x_0, -x_1, x_2, -x_3 \dots, x_{n-1}, -x_n)$. □

To see an example of degree 2, we will have to start with S^1. First, let us describe a generator of $\mathrm{H}_1^{\mathrm{Sing}}(S^1)$. We will think of S^1 as the unit circle in $\mathbb{C}$, and of Δ^1 as the interval $I = [0,1]$. Let $\nu, \sigma : I \to S^1$ be given by

$$\nu(t) = e^{i\pi t}$$
$$\sigma(t) = e^{-i\pi t}$$

ν, σ define 1-simplicies in $\mathrm{Sing}(S^1)$, and $\nu - \sigma$ is a cycle.



Proposition 2.7. $[\nu - \sigma]$ *is a generator of* $\mathrm{H}_1^{\mathrm{Sing}}(S^1)$.

Proof. Consider reduced Mayer-Vietoris for the covering $S^1 = N \cup S$, with $N \cap S = E \simeq S^0$.

$$0 = \widetilde{\mathrm{H}}_1^{\mathrm{Sing}}(N) \oplus \widetilde{\mathrm{H}}_1^{\mathrm{Sing}}(S) \to \widetilde{\mathrm{H}}_1^{\mathrm{Sing}}(S^n) \xrightarrow{d} \widetilde{\mathrm{H}}_0^{\mathrm{Sing}}(E) \to \widetilde{\mathrm{H}}_0^{\mathrm{Sing}}(N) \oplus \widetilde{\mathrm{H}}_0^{\mathrm{Sing}}(S) = 0$$

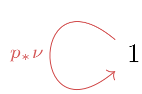
In particular, d is an isomorphism. Note that $[\nu - \sigma]$ is already split into a part in N and a part in S, so by definition

$$d([\nu - \sigma]) = \partial\nu = \partial\sigma = [-1] - [1]$$

and we saw that $[-1] - [1]$ is a generator of $\widetilde{\mathrm{H}}_0^{\mathrm{Sing}}(E)$. □

Proposition 2.8. *Consider* S^1 *as the unit circle in* $\mathbb{C}$*. Then* $p \colon S^1 \to S^1$ *given by* $p(z) = z^2$ *has degree 2.*

Proof. After squaring, $p_*\nu(t) = e^{2\pi i t}$ is the path going from 1 to itself counter-clockwise:



In particular $[p_*\nu] = [\nu - \sigma]$. Similarly, $p_*\sigma(t) = e^{-2\pi i t}$ goes from 1 to itself clockwise, so $[p_*\sigma] = -[\nu - \sigma]$. It follows that $p_*[\nu - \sigma] = 2[\nu - \sigma]$. ☐

There is another way to see that $\deg(p) = 2$. Consider S^1 as pointed by $1 \in S^1$. Notice that $p(-1) = 1$ and so p factors through the quotient $S^1/\{-1, 1\} \to S^1$. Note moreover that $S^1/\{-1, 1\} \simeq S^1 \vee S^1$. Thus, p factors as a map $S^1 \to S^1 \vee S^1 \to S^1$, where the first map pinches S^1 in the middle and the second map sends both copies through the identity.

$$\begin{array}{ccccc} S^1 & & \vee & & S^1 \\ & \overset{\text{id}}{\searrow} & & \overset{\text{id}}{\swarrow} & \\ & & S^1 & & \end{array}$$

On homology, this induces

$$\mathrm{H}_1^{\mathrm{Sing}}(S^1) \to \mathrm{H}_1^{\mathrm{Sing}}(S^1) \oplus \mathrm{H}_1^{\mathrm{Sing}}(S^1) \to \mathrm{H}_1^{\mathrm{Sing}}(S^1)$$

The first map sends a generator of $\mathrm{H}_1^{\mathrm{Sing}}(S^1)$ to the same generator in both pinched circles $1 \mapsto (1, 1)$, and the second map adds the two coordinates $(1, 1) \mapsto 2$.

We used the following fact about the homology of wedge:

Proposition 2.9. *Let $X, Y \in \mathrm{Top}_*$, such that $* \in X$ and $* \in Y$ have a contractible neighborhood, then for all $n \geq 0$ the assembly map*

$$\widetilde{\mathrm{H}}_n^{\mathrm{Sing}}(X) \oplus \widetilde{\mathrm{H}}_n^{\mathrm{Sing}}(Y) \to \widetilde{\mathrm{H}}_n^{\mathrm{Sing}}(X \vee Y)$$

is an isomorphism.

Proof. By the existence of a contractible neighborhood, we can use X, Y as a covering of $X \vee Y$ for Mayer-Vietoris, with intersection $X \cap Y = \mathrm{pt}$.

$$0 = \widetilde{\mathrm{H}}_n^{\mathrm{Sing}}(\mathrm{pt}) \to \widetilde{\mathrm{H}}_n^{\mathrm{Sing}}(X) \oplus \widetilde{\mathrm{H}}_n^{\mathrm{Sing}}(Y) \to \widetilde{\mathrm{H}}_n^{\mathrm{Sing}}(X \vee Y) \to \widetilde{\mathrm{H}}_n^{\mathrm{Sing}}(\mathrm{pt}) = 0,$$

so the map in the middle is an isomorphism. Verify that this map is indeed the assembly map. ☐

We can use p to define inductively a map $S^n \to S^n$ with degree 2. For that, we will use suspensions.

Definition 2.10. Let $\Sigma : \mathrm{Top} \to \mathrm{Top}$ be the functor sending $X \in \mathrm{Top}$ to the pushout

$$\begin{array}{ccc} X \times \{-1, 1\} & \longrightarrow & X \times [-1, 1] \\ \downarrow & & \downarrow \\ \{-1, 1\} & \longrightarrow & \Sigma X \end{array}$$

Explicitly, $\Sigma X = X \times [-1, 1]/\sim$ where the equivalence relation identifies all points in $X \times \{-1\}$ and all point in $X \times \{1\}$ separetley. A map $f \colon X \to Y$ is sent to $\Sigma f \colon \Sigma X \to \Sigma Y$, given by $[(x, t)] \mapsto [(f(x), t)]$.

Example 2.11. $\Sigma S^{n-1} \simeq S^n$. with homeomorphism given by

$$((x_0, \dots, x_{n-1}), t) \mapsto (\sqrt{1-t^2}x_0, \dots, \sqrt{1-t^2}x_{n-1}, t)$$

In particular, if $r : S^{n-1} \to S^{n-1}$ is the reflection of the first coordinate, then Σr is also the reflection of the first coordinate.

Proposition 2.12. *Let* $f\colon S^{n-1} \to S^{n-1}$, *then* $\deg(\Sigma f) = \deg(f)$.

Proof. This is the same idea as in the special case of reflection. Consider the covering $S^n = N \cup S$, such that under the identification $S^n = \Sigma S^{n-1}$, N corresponds to $S^{n-1} \times [0,1]$ and S corresponds to $S^{n-1} \times [-1,0]$. In particular, $\Sigma f(N) = N$, $\Sigma f(S) = S$. Note that $E = S^{n-1} \times \{0\}$, and $\Sigma f|_E = f$. By naturality of Mayer-Vietoris

$$\begin{array}{ccccccc}
0 = \widetilde{\mathrm{H}}_n^{\mathrm{Sing}}(N) \oplus \widetilde{\mathrm{H}}_n^{\mathrm{Sing}}(S) & \longrightarrow & \widetilde{\mathrm{H}}_n^{\mathrm{Sing}}(S^n) & \xrightarrow{\sim} & \widetilde{\mathrm{H}}_{n-1}^{\mathrm{Sing}}(E) & \longrightarrow & \widetilde{\mathrm{H}}_{n-1}^{\mathrm{Sing}}(N) \oplus \widetilde{\mathrm{H}}_{n-1}^{\mathrm{Sing}}(S) = 0 \\
 & & \downarrow{\scriptstyle \Sigma f_*} & & \downarrow{\scriptstyle f_*} & & \\
0 = \widetilde{\mathrm{H}}_n^{\mathrm{Sing}}(N) \oplus \widetilde{\mathrm{H}}_n^{\mathrm{Sing}}(S) & \longrightarrow & \widetilde{\mathrm{H}}_n^{\mathrm{Sing}}(S^n) & \xrightarrow{\sim} & \widetilde{\mathrm{H}}_{n-1}^{\mathrm{Sing}}(E) & \longrightarrow & \widetilde{\mathrm{H}}_{n-1}^{\mathrm{Sing}}(N) \oplus \widetilde{\mathrm{H}}_{n-1}^{\mathrm{Sing}}(S) = 0
\end{array}$$

so $\deg(\Sigma f) = \deg(f)$. □

Corollary 2.13. *Consider* $\Sigma^{n-1}p\colon S^n \to S^n$, *then* $\deg(\Sigma^{n-1}p) = 2$.