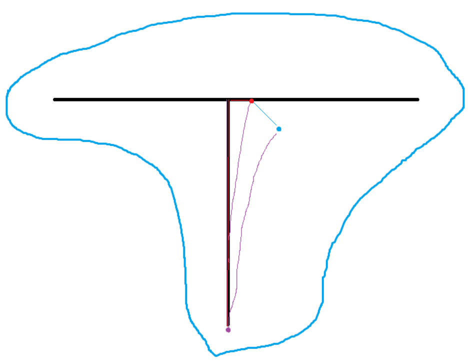
Algebraic topology - Recitation 5

December 2, 2024

1 Comments on homework

I will start by presenting common comments I had on homework 2.

exercise 1.1. Consider the house with two rooms. For ease of drawing, we will consider the simpler T shape T. As the hint suggested, we will look at an ϵ-neighborhood $T \subseteq N$. The first common error was in defining the retract $r\colon N \to T$ as sending each point in N to the nearest point in T. This operation is not continuous near the corners. Instead, we can take inspiration from physics — think of T as a heavy rigid mass, and N as a cloud around it that is gravitationally pulled. This will be a continuous deformation retract, due to the continuous nature of reality. We can make this formal (write the laws of motion), but we could also produce simpler equations based on this idea. The second step is to notice that N is homeomorphic to the disk, so it is contractible to a point. However, another common error is that a deformation retract on N does not restrict to a deformation retract on T, as the path a point $x \in T$ goes through does not have to be contained in T. To fix this, apply r to this path.



exercise 1.2. In the question named "long exact sequence in homology", you were asked to prove

the nine lemma. Consider the commuting diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_3 & \longrightarrow & B_3 & \longrightarrow & C_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_2 & \longrightarrow & B_2 & \longrightarrow & C_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_1 & \longrightarrow & B_1 & \longrightarrow & C_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where the rows and the middle column are exact. Then the first column is exact if and only if the last column is exact. Almost all of you proved it by diagram chasing, which is a great exercise, but not the most efficient solution. Instead, start by only assuming that the middle column is a chain complex (with 0's extended in both direction), meaning that the composition is 0. It follows by the exactness of the rows that the first and third column are also chain complexes:

- Any $c \in C_3$ can be lifted by surjectivity to $b \in B_3$, which is mapped to $0 \in B_1$, which is mapped to 0 in C_1.
- Any $a \in A_3$ is mapped to 0 in B_1, so by injectivity it is also mapped to 0 in A_1.

So we have an exact sequence of chain complexes

$$0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0.$$

Note that a chain complex $B_\bullet$ is exact iff $\mathrm{H}_n(B_\bullet) = 0$ for all $n \in \mathbb{Z}$. Thus, assuming that $B_\bullet$ is exact, we get a long exact sequence in homology

$$\dots 0 \to H_n(A_\bullet) \to H_{n-1}(C_\bullet) \to 0 \dots$$

and in particular $A_\bullet$ is exact if and only if $C_\bullet$ is exact.

2 CW complexes

A CW-complex, or cellular complex, is a space that is built by iteratively gluing disks along their boundary.

Definition 2.1. A *cell structure* on a space X is a filtration

$$X^0 \subseteq X^1 \subseteq X^2 \subseteq \dots X$$

where X^n is called the *n-skeleton*, together with a set of *n-cells* $\{\Phi^n_\alpha \colon D^n \to X^n\}_{\alpha \in I_n}$ for $n \geq 0$, such that:

(1) The boundary of every n-cell is mapped to the $n-1$-skeleton $\varphi^n_\alpha := \Phi^n_\alpha|_{S^{n-1}} \colon S^{n-1} \to X^{n-1}$ (where $S^{-1} = X^{-1} = \varnothing$). φ^n_α are called the *attaching maps*.

(2) X^n is produced from X^{n-1} by gluing the n-cells along their boundary $S^{n-1} \hookrightarrow D^n$ via the attaching maps:

$$\begin{array}{ccc} \coprod_{\alpha \in I_n} S^{n-1} & \hookrightarrow & \coprod_{\alpha \in I_n} D^n \\ \downarrow{\scriptstyle [\varphi^n_\alpha]_{\alpha \in I_n}} & & \downarrow{\scriptstyle [\Phi^n_\alpha]_{\alpha \in I_n}} \\ X^{n-1} & \hookrightarrow & X^n \end{array}$$

(3) $X = \bigcup_{n<\infty} X_n$

A *CW complex* is a space which has a cell structure. Note that to define the cell structure we actually need the cells, as the skeletons are built from the cells, but the conditions are easier to state this way.

For $n = 0$, as $\coprod_{\alpha \in I_n} S^{n-1} = X^{-1} = \varnothing$, we get $X^0 = \coprod_{\alpha \in I_0} D^0$, i.e. a discrete space of points, without gluing data. If X is n-dimensional, i.e. doesn't have any cells higher than n, then $X^n = X^{n+1} = \cdots = X$, and we will write the filtration only up to level n.

Example 2.2. There are multiple cell structures on S^n. The standard one consists of a single 0-cell and a single n-cell, such that the filtration is given by

$$\mathrm{pt} = \mathrm{pt} = \cdots = \mathrm{pt} \subseteq S^n$$

and the n-cell is the map $\Phi \colon D^n \to S^n$ which collapses the boundary to the point $\varphi \colon S^{n-1} \to \mathrm{pt}$. Indeed, we have the pushout

$$\begin{array}{ccc} S^{n-1} & \hookrightarrow & D^n \\ \downarrow & & \downarrow \\ \mathrm{pt} & \hookrightarrow & D^n/S^{n-1} \simeq S^n \end{array}$$

Example 2.3. For another cell structure on S^n, consider the inclusion of the equator $S^{k-1} \subseteq S^k$. This produces a filtration

$$S^0 \subseteq S^1 \subseteq S^2 \subseteq \cdots \subseteq S^n.$$

Notice that each S^k is produced from gluing the northern and southern hemispheres $\Phi^k_N, \Phi^k_S \colon D^k \to S^k$ along the equator $\varphi^k_N = \varphi^k_S = \mathrm{id}_{S^{k-1}}$ (S^0 is just two points without gluing). That is, we have a pushout square

$$\begin{array}{ccc} S^{k-1} \sqcup S^{k-1} & \hookrightarrow & D^n \sqcup D^n \\ \downarrow{\scriptstyle [\mathrm{id}_{S^{k-1}}, \mathrm{id}_{S^{k-1}}]} & & \downarrow \\ S^{k-1} & \hookrightarrow & S^k \end{array}$$

Thus, we get a cell structure consisting of 2 k-cells for every $0 \leq k \leq n$.

Example 2.4. Consider $\mathbb{RP}^n$, the space of lines in $\mathbb{R}^{n+1}$, or equivalently $S^n/{-x} \sim x$. The equator inclusion $S^{k-1} \subseteq S^k$ induce inclusions $\mathbb{RP}^{k-1} \subseteq \mathbb{RP}^k$, which defines a filtration

$$\mathbb{RP}^0 \subseteq \mathbb{RP}^1 \subseteq \mathbb{RP}^2 \subseteq ... \subseteq \mathbb{RP}^n.$$

In each dimension $0 \leq k \leq n$ we have a single k-cell

$$\Phi^k \colon D^k \xhookrightarrow{\Phi_N^k} S^k \xrightarrow{q} \mathbb{RP}^k.$$

On the boundary S^{k-1} we just take the quotient $-x \sim x$ and produce $\mathbb{RP}^{k-1}$, so $\varphi^k = q \colon S^{k-1} \to \mathbb{RP}^{k-1}$. Moreover, for any $[x] \in \mathbb{RP}^k$ we could choose the representative $x \in S^k$ to come from the northern hemisphere, and this representative is unique in the interior. Thus, $\mathbb{RP}^k$ is produced from $\mathbb{RP}^{k-1}$ by attaching a single k-cell along the quotient map

$$\begin{array}{ccc} S^{k-1} & \hookrightarrow & D^n \\ \downarrow{\scriptstyle q} & & \downarrow \\ \mathbb{RP}^{k-1} & \hookrightarrow & \mathbb{RP}^k \end{array}$$

In the exercise you are supposed to use this filtration on $\mathbb{RP}^n$ to compute its homology using Mayer-Vietoris. Soon we will generalize this method to calculate the homology of arbitrary CW-complexes. Notice that while CW-complexes are somewhat similar to semisimplicial spaces, they are usually much easier to define, because we don't have to triangulate anything.

In the rest of the recitation we will gather more examples of CW-complexes, so we will have things to calculate later.

Example 2.5. There are examples of CW-complexes with infinite dimension. In the exercise you saw the infinite sphere

$$S^0 \subseteq S^1 \subseteq \cdots \subseteq S^\infty = \{(x_0, x_1, \dots) | \sum_i x_i^2 = 1,\ x_i = 0 \text{ for all but finitely many } i\}.$$

This cell structure has 2 cells in each dimension. Define $\mathbb{RP}^\infty = S^\infty / -x \sim x$, this has a cellular structure with 1 cell in each dimension

$$\mathbb{RP}^0 \subseteq \mathbb{RP}^1 \subseteq \cdots \subseteq \mathbb{RP}^\infty.$$

Example 2.6. The complex projective space $\mathbb{CP}^n$ is the topological space of 1-dimensional subspaces (complex lines) in $\mathbb{C}^{n+1}$. Alternatively, the unit sphere in $\mathbb{C}^{n+1}$ is identified with S^{2n+1}, and we define $\mathbb{CP}^n = S^{2n+1}/\forall\lambda \in U(1)\ \lambda x \sim x$ with the quotient topology ($U(1)$ is the group of unit vectors in $\mathbb{C}$, which acts on S^{2n+1}). This quotient comes with a quotient map $q \colon S^{2n+1} \to \mathbb{CP}^n$. To define the cell structure on $\mathbb{CP}^n$, we will describe how $\mathbb{CP}^n$ is constructed from $\mathbb{CP}^{n-1}$ by attaching a single $2n$-cell. Consider the inclusion $D^{2n} \hookrightarrow S^{2n+1}$, where D^{2n} is considered as the unit ball in $\mathbb{C}^n$, given by Consider the subspace

$$(z_0, \dots, z_{n-1}) \mapsto (z_0, \dots, z_{n-1}, \sqrt{1 - (|z_0|^2 + \cdots + |z_{n-1}|^2)}).$$

The image of this inclusion is the subspaces

$$R = \{z = (z_0, \dots, z_n) \in S^{2n+1} \mid z_n \text{ is real and non-negative}\} \subseteq S^{2n+1}.$$

The $2n$-cell of $\mathbb{CP}^n$ is given by

$$\Phi \colon D^{2n} \hookrightarrow S^{2n+1} \xrightarrow{q} \mathbb{CP}^n.$$

The boundary S^{2n-1} is sent by the inclusion to $\{(z_0, \dots, z_{n-1}, 0)\}$, which is mapped by q to $\mathbb{CP}^{n-1} \subseteq \mathbb{CP}^n$. Moreover, for any $[z] \in \mathbb{CP}^n$ we may assume $z \in R$ by rotating until z_n is a non-negative real (if $z_n = re^{i\theta}$, multiply by $e^{-i\theta}$), and on the interior there is only a single representative from R. Thus $\mathbb{CP}^n$ is produced from $\mathbb{CP}^{n-1}$ by attaching a single $2n$-cell along the quotient map

$$\begin{array}{ccc} S^{2n-1} & \hookrightarrow & D^{2n} \\ \downarrow q & \ulcorner & \downarrow \\ \mathbb{CP}^{n-1} & \hookrightarrow & \mathbb{RP}^n \end{array}$$

Taking the union over all n, we get $\mathbb{CP}^\infty$, which has a single cell in every even degree.

Example 2.7. We saw many constructions given by identifying edges on a polygon, those are actually CW complexes. Consider the surface of genus Σ_g. Cutting between the holes, we see that Σ_g can be formed by gluing tori with 2 holes cut out along their holes, with the edges having only a single. This is equivalent to taking a polygon with $4g$ and identifying the edges according to

$$a_1, b_1, a_1, b_1, a_2, b_2, a_2, b_2, \dots, a_g, b_g, a_g, b_g$$

To interpret this as a cellular complex, notice that all vertices of this polygon are identified to be the same point, and there are $2g$ distinct edges after identification. Thus, we can get a cell structure with a single 0-cell, $2g$ 1-cells and a single 2-cell

$$\mathrm{pt} \subseteq \bigvee_{i=1}^{g} S^1_{a_i} \vee S^1_{b_i} \subseteq \Sigma_g$$

and the attaching map of the 2-cell $\varphi\colon S^1 \to \bigvee_{i=1}^{g} S^1_{a_i} \vee S^1_{b_i}$ follows the path described above.