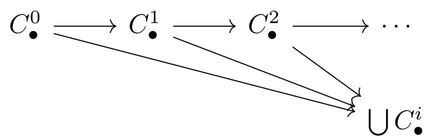
Algebraic topology - Recitation 6

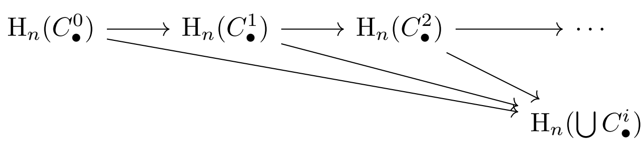
December 9, 2024

1 Comments on homework

exercise 1.1. In the exercise you were asked to construct an isomorphism $\mathrm{H}_n(\bigcup C^i_\bullet) \simeq \varinjlim \mathrm{H}_n(C^i_\bullet)$. Most of you constructed the map from left to right, however it is always easier to map out of a colimit, especially when you proved the universal property is part (a). Moreover, the union $\bigcup C^i_\bullet$ is a directed limit in Ch, so the map will $\varinjlim \mathrm{H}_n(C^i_\bullet) \to \mathrm{H}_n(\bigcup C^i_\bullet)$ will be the assembly map. Explicitly, the following diagram of inclusions commutes



so by functoriality of H_n, the following diagram also commutes



By the universal property there is a map $\varinjlim \mathrm{H}_n(C^i_\bullet) \to \mathrm{H}_n(\bigcup C^i_\bullet)$, and it remains to show that it is an isomorphism. Equivalently, one can show that the diagram with $\mathrm{H}_n(\bigcup C^i_\bullet)$ satisfies the universal property.

In the same exercise, you were asked to prove that S^∞ is contractible. This does not follow from the calculation of its homology, as there exists non-contractible spaces which have the homology of a point. Such spaces are called acyclic. The trick here was to use the infinite space available to shift all coordinates one space right, leaving the first coordinate 0. Then we can increase the first coordinate and decrease the rest, to contract to $(1, 0, 0, \dots)$.

2 A teaser for cellular homology

CW-complexes are a well-behaved example of a filtered space, given by the skeleton filtration $X_0 \subseteq X_1 \subseteq X_2 \dots$. In the lecture you are learning a general procedure to calculate the homology of a filtered space, and we will later use it for CW complexes. To get a feel for this procedure, we will do the case of $\mathbb{RP}^n$.

Proposition 2.1. *The homology of $\mathbb{RP}^n$ is given by*

$$\widetilde{\mathrm{H}}_k^{\mathrm{Sing}}(\mathbb{RP}^n) = \begin{cases} \mathbb{Z} & k = n \text{ for } n \text{ odd} \\ \mathbb{Z}/2\mathbb{Z} & 0 < k < n \text{ odd} \\ 0 & \text{else} \end{cases}$$

Moreover, for n odd, the map $q_* \colon \widetilde{\mathrm{H}}_n^{\mathrm{Sing}}(S^n) \to \widetilde{\mathrm{H}}_n^{\mathrm{Sing}}(\mathbb{RP}^n)$ corresponds to $\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}$.

Proof. We will proof by induction on n. Consider the CW-pair $(\mathbb{RP}^n, \mathbb{RP}^{n-1})$. $\mathbb{RP}^n$ is built from $\mathbb{RP}^{n-1}$ by gluing a single n-cell along the boundary. If we stabilize $\mathbb{RP}^{n-1}$ to a point, then we will remain with a single n-cell with its boundary glued to the point, which will result in $\mathbb{RP}^n/\mathbb{RP}^{n-1} \simeq S^n$. This follows from the exercise in the homework on quotients of CW-pairs, or from the pushout

$$\begin{array}{ccc} S^{n-1} & \longrightarrow & D^n \\ \downarrow & & \downarrow \\ \mathbb{RP}^{n-1} & \longrightarrow & \mathbb{RP}^n \\ \downarrow & & \downarrow \\ \mathrm{pt} & \longrightarrow & S^n \end{array}$$

Thus, we get a long exact sequence in homology

$$\cdots \to \widetilde{\mathrm{H}}_{k+1}^{\mathrm{Sing}}(S^n) \to \widetilde{\mathrm{H}}_k^{\mathrm{Sing}}(\mathbb{RP}^{n-1}) \to \widetilde{\mathrm{H}}_k^{\mathrm{Sing}}(\mathbb{RP}^n) \to \widetilde{\mathrm{H}}_k^{\mathrm{Sing}}(S^n) \to \dots$$

For $k \neq n, n-1$ this implies $\widetilde{\mathrm{H}}_k^{\mathrm{Sing}}(\mathbb{RP}^{n-1}) \simeq \widetilde{\mathrm{H}}_k^{\mathrm{Sing}}(\mathbb{RP}^n)$. For $k = n, n-1$, consider the CW-pair (S^n, S^{n-1}). The quotient map $q \colon S^n \to \mathbb{RP}^n$ sends S^{n-1} to $\mathbb{RP}^{n-1}$, so it induces a map of CW-pairs

$$q \colon (S^n, S^{n-1}) \to (\mathbb{RP}^n, \mathbb{RP}^{n-1}),$$

so by naturality

$$\widetilde{\mathrm{H}}_n^{\mathrm{Sing}}(\mathbb{RP}^{n-1}) \to \widetilde{\mathrm{H}}_n^{\mathrm{Sing}}(\mathbb{RP}^n) \to \widetilde{\mathrm{H}}_n^{\mathrm{Sing}}(S^n) \to \widetilde{\mathrm{H}}_{n-1}^{\mathrm{Sing}}(\mathbb{RP}^{n-1}) \to \widetilde{\mathrm{H}}_{n-1}^{\mathrm{Sing}}(\mathbb{RP}^n) \to \widetilde{\mathrm{H}}_{n-1}^{\mathrm{Sing}}(S^n)$$

and split into cases.

- If n is odd, then we get

  $$0 \to \widetilde{\mathrm{H}}_n^{\mathrm{Sing}}(\mathbb{RP}^n) \to \mathbb{Z} \to 0 \to \widetilde{\mathrm{H}}_{n-1}^{\mathrm{Sing}}(\mathbb{RP}^n) \to 0$$

  so $\widetilde{\mathrm{H}}_{n-1}^{\mathrm{Sing}}(\mathbb{RP}^n) = 0$ and $\widetilde{\mathrm{H}}_n^{\mathrm{Sing}}(\mathbb{RP}^n) \simeq \mathbb{Z}$. To show that $q_* \colon \widetilde{\mathrm{H}}_n^{\mathrm{Sing}}(S^n) \to \widetilde{\mathrm{H}}_n^{\mathrm{Sing}}(\mathbb{RP}^n)$ corresponds to $\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}$, consider the CW-pair (S^n, S^{n-1}). The quotient map $q \colon S^n \to \mathbb{RP}^n$ sends S^{n-1} to $\mathbb{RP}^{n-1}$, so it induces a map of CW-pairs

  $$q \colon (S^n, S^{n-1}) \to (\mathbb{RP}^n, \mathbb{RP}^{n-1}),$$

and by naturality

$$\begin{array}{ccccccc} & & \widetilde{\mathrm{H}}_n^{\mathrm{Sing}}(S^n) & \longrightarrow & \widetilde{\mathrm{H}}_n^{\mathrm{Sing}}(S^n/S^{n-1}) & & \\ & & \downarrow{\scriptstyle q_*} & & \downarrow & & \\ 0 & \longrightarrow & \widetilde{\mathrm{H}}_n^{\mathrm{Sing}}(\mathbb{RP}^n) & \xrightarrow{\sim} & \widetilde{\mathrm{H}}_n^{\mathrm{Sing}}(S^n) & \longrightarrow & 0 \end{array}$$

To find q^*, we need to find the composition along the top and right. Note that $S^n/S^{n-1} \simeq S^n \vee S^n$, where the two components correspond to the northren and southren hemispheres, and the map $S^n \vee S^n \to S^n$ is induced by the identity of the northern hemisphere and the reflection of the southern hemisphere. Thus, the composition along the top and right is induced from

$$S^n \to S^n/S^{n-1} \xrightarrow{\sim} S^n \vee S^n \xrightarrow{[\mathrm{id}, -\mathrm{id}]} S^n,$$

and in the exercise you saw that the degree of this map is $\deg(\mathrm{id}) + \deg(-\mathrm{id}) = 1 + (-1)^{n+1} = 2$.

- If n is odd, then we get

$$0 \to \widetilde{\mathrm{H}}_n^{\mathrm{Sing}}(\mathbb{RP}^n) \to \widetilde{\mathrm{H}}_n^{\mathrm{Sing}}(S^n) \xrightarrow{d} \widetilde{\mathrm{H}}_{n-1}^{\mathrm{Sing}}(\mathbb{RP}^{n-1}) \to \widetilde{\mathrm{H}}_{n-1}^{\mathrm{Sing}}(\mathbb{RP}^n) \to 0,$$

and we want to find d. This time, we will use the two maps of pairs

$$(D^n, S^{n-1}) \hookrightarrow (S^n, S^{n-1}) \xrightarrow{q} (\mathbb{RP}^n, \mathbb{RP}^{n-1})$$

where the first map is the inclusion of the northern hemisphere. By naturality, we get

$$\begin{array}{ccccccc} \widetilde{\mathrm{H}}_n^{\mathrm{Sing}}(D^n) & \longrightarrow & \widetilde{\mathrm{H}}_n^{\mathrm{Sing}}(D^n/S^{n-1}) & \longrightarrow & \widetilde{\mathrm{H}}_{n-1}^{\mathrm{Sing}}(S^{n-1}) & \longrightarrow & \widetilde{\mathrm{H}}_{n-1}^{\mathrm{Sing}}(D^n) \\ & & \downarrow & & \| & & \\ & & \widetilde{\mathrm{H}}_n^{\mathrm{Sing}}(S^n/S^{n-1}) & \longrightarrow & \widetilde{\mathrm{H}}_{n-1}^{\mathrm{Sing}}(S^{n-1}) & & \\ & & \downarrow & & \downarrow & & \\ & & \widetilde{\mathrm{H}}_n^{\mathrm{Sing}}(S^n) & \xrightarrow{d} & \widetilde{\mathrm{H}}_{n-1}^{\mathrm{Sing}}(\mathbb{RP}^{n-1}) & & \end{array}$$

Plugging in the known homology groups and maps, we get

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\sim} & \mathbb{Z} & \longrightarrow & 0 \\ & & \downarrow & & \| & & \\ & & \mathbb{Z} \oplus \mathbb{Z} & \longrightarrow & \mathbb{Z} & & \\ & & \downarrow & & {\scriptstyle \times 2}\downarrow & & \\ & & \mathbb{Z} & \xrightarrow{d} & \mathbb{Z} & & \end{array}$$

However, the left vertical maps are induced from

$$S^n \simeq D^n/S^{n-1} \hookrightarrow S^n/S^{n-1} \xrightarrow{q} \mathbb{RP}^n/\mathbb{RP}^{n-1} \simeq S^n$$

which composes to a homeomorphism. Thus, the composition of the left vertical maps is an isomorphism, and by commutativity of the above square we see that $d = \times 2$ (up to a sign). In particular, it follows that $\widetilde{\mathrm{H}}_n^{\mathrm{Sing}}(\mathbb{RP}^n) = \ker(\times 2) = 0$ and $\widetilde{\mathrm{H}}_{n-1}^{\mathrm{Sing}}(\mathbb{RP}^n) \simeq \mathbb{Z}/2\mathbb{Z}$.

□

3 Spectral sequences

3.1 Filtration of length 2

Consider a filtered chain complex of length 2

$$0 = C^{(2)} \subseteq C^{(1)} \subseteq C^{(0)} = C$$

The corresponding spectral sequence computes $\mathrm{H}_n(C)/\mathrm{H}_n^{(1)}(C)$ using $\mathrm{H}_n(C^{(1)})$ and $\mathrm{H}_n(C^{(0)}/C^{(1)})$. Let us write the different pages of the spectral sequence. The zero page is $E^0_{n,s} = C_n^{(s)}/C_n^{(s+1)}$:

$$\begin{array}{cccccccccc}
\cdots & 0 & & 0 & & 0 & & 0 & \cdots \\
\cdots & C_{-1}^{(1)} & \longleftarrow & C_0^{(1)} & \longleftarrow & C_1^{(1)} & \longleftarrow & C_2^{(1)} & \cdots \\
\cdots & C_{-1}^{(0)}/C_{-1}^{(1)} & \longleftarrow & C_0^{(0)}/C_0^{(1)} & \longleftarrow & C_1^{(0)}/C_1^{(1)} & \longleftarrow & C_2^{(0)}/C_2^{(1)} & \cdots \\
\cdots & 0 & & 0 & & 0 & & 0 & \cdots
\end{array}$$

The 1 page is $E^1_{n,s} = \mathrm{H}_n(C^(s)/C^{s+1})$:

$$\begin{array}{cccccc}
\cdots & 0 & 0 & 0 & 0 & \cdots \\
\cdots & \mathrm{H}_{-1}(C^{(1)}) & \mathrm{H}_0(C^{(1)}) & \mathrm{H}_1(C^{(1)}) & \mathrm{H}_2(C^{(1)}) & \cdots \\
 & \nwarrow^{d_0} & \nwarrow^{d_1} & \nwarrow^{d_2} & & \\
\cdots & \mathrm{H}_{-1}(C^{(0)}/C^{(1)}) & \mathrm{H}_0(C^{(0)}/C^{(1)}) & \mathrm{H}_1(C^{(0)}/C^{(1)}) & \mathrm{H}_2(C^{(0)}/C^{(1)}) & \cdots \\
\cdots & 0 & 0 & 0 & 0 & \cdots
\end{array}$$

At the 2-page we will have the kernel and cokernekl of d_n:

$$\begin{array}{cccccc}
\cdots & 0 & 0 & 0 & 0 & \cdots \\
\cdots & \mathrm{coker}(d_0) & \mathrm{coker}(d_1) & \mathrm{coker}(d_2) & \mathrm{coker}(d_3) & \cdots \\
\cdots & \ker(d_{-1}) & \ker(d_0) & \ker(d_2) & \ker(d_3) & \cdots \\
\cdots & 0 & 0 & 0 & 0 & \cdots
\end{array}$$

Notice that from this point on all differentials will be 0, so the next pages will be equal to the 2-page, and in particular

$$E^2_{n,s} = E^{\infty}_{n,s} = \mathrm{H}^{(s)}_n(C)/\mathrm{H}^{(s+1)}_n(C)$$

We say that the spectral sequence stabilizes at page 2. In particular, for $s = 0$ we get

$$\mathrm{H}^{(0)}_n(C)/\mathrm{H}^{(1)}_n(C) = \ker(d_n)$$

which fits into an exact sequence

$$\mathrm{H}_n(C^{(1)}) \to \mathrm{H}_n(C^{(0)}) \to \mathrm{H}_n(C^{(0)}/C^{(1)}) \xrightarrow{d_n} \mathrm{H}_{n-1}(C^{(1)})$$

continuing this sequence from both sides, we reconstructed the long exact sequence in homology.

From that perspective, spectral sequences for longer filtration are generalization of the long exact sequence in homology.

3.2 The stupid filtration

Let C be a positive chain complex, meaning $C_k = 0$ for $k < 0$. We can always define the stupid filtration on C,

$$C = C^{(0)} \geq C^{(1)} \geq C^{(2)} \geq \dots$$

given by

$$C^{(s)}_n = \begin{cases} C_n & n > s \\ \ker(\partial_s) & n = s \\ 0 & n < s \end{cases}.$$

Note that we had to replace C_s with the kernel, so that $C^{(s+1)} \leq C^{(s)}$ will be an inclusion of chain complexes

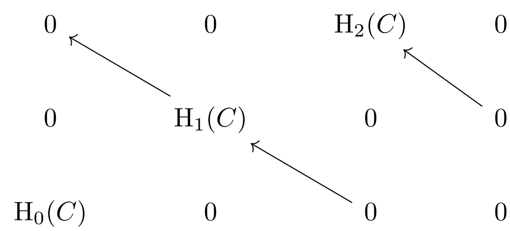
$$\begin{array}{ccccccc} 0 & \longleftarrow & 0 & \longleftarrow & \ker(\partial_{s+1}) & \xleftarrow{\partial_{s+2}} & C_{n+2} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \xleftarrow{\partial_n} & \ker(\partial_s) & \xleftarrow{\partial_{s+1}} & C_{s+1} & \xleftarrow{\partial_{s+2}} & C_{n+2} \end{array}$$

The associated graded is

$$E^0_{n,s} = C^{(s)}_n/C^{(s+1)}_n = \begin{cases} C_{s+1}/\ker(\partial_{s+1}) \simeq \mathrm{Im}(\partial_{s+1}) & n = s+1 \\ \ker(\partial_s) & n = s \\ 0 & n \neq s, s+1 \end{cases}$$

$$\begin{array}{ccccccc} 0 & & 0 & & \ker(\partial_2) & \longleftarrow & \mathrm{Im}(\partial_3) \\ \\ 0 & & \ker(\partial_1) & \longleftarrow & \mathrm{Im}(\partial_2) & & 0 \\ \\ \ker(\partial_0) & \longleftarrow & \mathrm{Im}(\partial_1) & & 0 & & 0 \end{array}$$

The 1-page is then



(we could have also started with the 1-page). The spectral sequence stabilizes after the 1-page. Thus, we learn that

$$\mathrm{H}_n^{(n)}(C)/\mathrm{H}_n^{(n+1)}(C) \simeq \mathrm{H}_n(C).$$

It is also easy to verify directly that $\mathrm{H}_n^{(n)}(C) = \mathrm{H}_n(C)$ and $\mathrm{H}_n^{(n+1)}(C) = 0$.

The stupid filtration is usually not very helpful, because the filtration has no geometric interpretation, as the chain complexes are not a homotopy invariant. However, in some cases it does have a geometric interpretation, and then it is surprisingly useful (the stupid filtration in Hodge theory).