Algebraic topology - Recitation 7

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1 Bicomplexes

Many important filtered complexes, on which we run spectral sequences, arise from bicomplexes.

Definition 1.1. A *bicomplex* is (approximately) a complex of chain complexes. Explicitly, it is a $\mathbb{Z} \times \mathbb{Z}$-diagram of abelian groups

$$\begin{array}{ccccc} B_{-1,1} & \xleftarrow{\partial^{<}_{0,1}} & B_{0,1} & \xleftarrow{\partial^{<}_{1,1}} & B_{1,1} \\ \downarrow{\scriptstyle \partial^{\vee}_{-1,1}} & & \downarrow{\scriptstyle \partial^{\vee}_{0,1}} & & \downarrow{\scriptstyle \partial^{\vee}_{1,1}} \\ B_{-1,0} & \xleftarrow{\partial^{<}_{0,0}} & B_{0,0} & \xleftarrow{\partial^{<}_{1,0}} & B_{1,0} \\ \downarrow{\scriptstyle \partial^{\vee}_{-1,0}} & & \downarrow{\scriptstyle \partial^{\vee}_{0,0}} & & \downarrow{\scriptstyle \partial^{\vee}_{1,0}} \\ B_{-1,-1} & \partial^{<}_{0,-1} & B_{0,-1} & \partial^{<}_{1,-1} & B_{1,-1} \end{array}$$

such that

(1) $\partial^{<}_{p,q}\partial^{<}_{p+1,q} = 0$

(2) $\partial^{\vee}_{p,q}\partial^{\vee}_{p,q+1} = 0$

(3) The obvious condition will be that each square commutes. However, to avoid later book-keeping, we will assert that every square anticommutes $\partial^{<}_{p+1,q}\partial^{\vee}_{p+1,q+1} = -\partial^{\vee}_{p,q+1}\partial^{<}_{p+1,q+1}$. A commuting bicomplex can be modified to an anticommuting bicomplex, e.g. by negating every even horizontal map.

Given a bicomplex, we can construct a chain complex called the *total complex*

$$\mathrm{Tot}(B)_n = \bigoplus_{p+q=n} B_{p,q}$$

where the boundary

$$\begin{aligned} \partial \colon \mathrm{Tot}(B)_n &\to \mathrm{Tot}(B)_{n-1} \\ \bigoplus_{p+q=n} B_{p,q} &\to \bigoplus_{p'+q'=n-1} B_{p',q'} \end{aligned}$$

is given by $\partial^{<} + \partial^{\vee}$ in every coordinate. Indeed, $\partial^2 = (\partial^{<} + \partial^{\vee})^2 = {\partial^{<}}^2 + \partial^{<}\partial^{\vee} + \partial^{\vee}\partial^{<} + {\partial^{\vee}}^2 = 0$. The total complex then has two natural increasing filtration:

- the *vertical filtration* $\mathrm{Tot}(B)_n^{\vee(s)} = \bigoplus_{p \le s} B_{p,n-p}$, and
- the *horizontal filtration* $\mathrm{Tot}(B)_n^{<(s)} = \bigoplus_{q \le s} B_{n-q,p}$.

We could transform them to a decreasing filtration by switching s with $-s$, but it is also fine to work with increasing filtration; the differentials simply go down instead of up.

From now on assume that $B_{p,q}$ is non-zero only is a bounded part of $\mathbb{Z}^2$, so we won't have convergence issues. The vertical and horizontal filtrations will produce different spectral sequences, but they can both be used to calculate the same homology $\mathrm{H}_n(\mathrm{Tot}(B))$. In fact, it is often the case that $\mathrm{H}_n(\mathrm{Tot}(B))$ is easy to calculate with one filtration, and we use this to figure something out about the other filtration.

To be more explicit, let us describe the 0 and 1 pages in both filtrations. For the vertical filtration, the associated graded is

$$E_{n,s}^{\vee 0} = \bigoplus_{p \le s} B_{p,n-p} / \bigoplus_{p \le s-1} B_{p,n-p} = B_{s,n-s}$$

with the differerntial coming from the vertical boundary $\partial^\vee \colon B_{s,n-s} \to B_{s,n-1-s}$ Thus the 1-page is the vertical homology $E_{n,s}^{\vee 1} = \mathrm{H}_n(B_{s,\bullet - s}) = \mathrm{H}_{n-s}(B_{s,\bullet})$. Similarly, the associated graded of the horizontal filtration is $B_{n-s,s}$ and the 1-page is the horizontal homology $E_{n,s}^{<1} = \mathrm{H}_{n-s}(B_{\bullet,s})$.

Example 1.2 (The snake lemma)**.** Consider the following bicomplex:

$$\begin{array}{ccccccccc}
0 & & 0 & & 0 & & 0 & & 0 \\
 & & & & & & & & \\
0 & \longleftarrow & C & \longleftarrow & B & \longleftarrow & A & \longleftarrow & 0 \\
 & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle -\beta} & & \downarrow{\scriptstyle \alpha} & & \\
0 & \longleftarrow & F & \longleftarrow & E & \longleftarrow & D & \longleftarrow & 0 \\
 & & & & & & & & \\
0 & & 0 & & 0 & & 0 & & 0
\end{array}$$

such that the rows are exact and the squares anticommute (they commute when we have β instead of $-\beta$), where F is in the $(0,0)$ coordinate. As all rows are exact, the horizontal homology will be 0, so the horizontal spectral sequence stabilizes at the 1-page to 0. In particular, $E_{n,s}^{<\infty} = 0$, which is the associated graded of some filtration on $\mathrm{H}_n(\mathrm{Tot}(B))$, so $\mathrm{H}_n(\mathrm{Tot}(B)) = 0$. Over to the vertical filtration, let us start with the 0-page $E_{n,s}^{\vee 0} = B_{s,n-s}$:

$$\begin{array}{ccccccc}
0 & & 0 & & D & \xleftarrow{\alpha} & A \\
 & & & & & & \\
0 & & E & \xleftarrow{-\beta} & B & & 0 \\
 & & & & & & \\
F & \xleftarrow{\gamma} & C & & 0 & & 0
\end{array}$$

The 1-page is then

$$\begin{array}{ccccccc} 0 & & 0 & & \operatorname{coker}(\alpha) & & \ker(\alpha) \\ & & & \swarrow & & \swarrow & \\ 0 & & \operatorname{coker}(\beta) & & \ker(\beta) & & 0 \\ & \swarrow & & \swarrow & & & \\ \operatorname{coker}(\gamma) & & \ker(\gamma) & & 0 & & 0 \end{array}$$

where the arrows go down in the increasing filtration. The 2-page then has the form

$$\begin{array}{cccc} 0 & 0 & X & ? \\ 0 & ? & ? & 0 \\ ? & Y & 0 & 0 \end{array} \qquad (X \to Y)$$

where the differentials at the ? squares will always hit 0, so the ? squares stabilize to the ∞-page. However, we already know that $\mathrm{H}_n(\mathrm{Tot}(B)) = 0$, so the ∞-page is 0, so $? = 0$. Similarly, from the 3-rd page onwards the X, Y squares stabilize at 0, so the map $X \to Y$ has to be an isomorphism. However X is the kernel of $\operatorname{coker}(\alpha) \to \operatorname{coker}(\beta)$ and Y is the cokernel of $\ker(\beta) \to \ker(\gamma)$, in particular we get a map

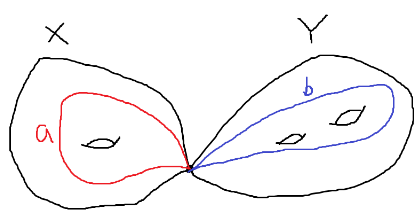
$$\ker(\gamma) \twoheadrightarrow Y \xleftarrow{\sim} X \hookrightarrow \operatorname{coker}(\alpha)$$

whose composition we will call $d\colon \ker(\gamma) \hookrightarrow \operatorname{coker}(\alpha)$. This map fits in an exact sequence

$$\ker(\alpha) \to \ker(\beta) \to \ker(\gamma) \xrightarrow{d} \operatorname{coker}(\alpha) \to \operatorname{coker}(\beta) \to \operatorname{coker}(\gamma).$$

2 Non-Abelian groups

Suppose X, Y are pointed spaces, and consider the wedge $X \vee Y$:



Every loop in $X \vee Y$ can be decomposed into a composition of loops in X and loops in Y, so the two subgroups $\pi_1(X), \pi_1(Y) \leq \pi_1(X \vee Y)$ generate $\pi_1(X \vee Y)$. Suppose $a \in \pi_1(X)$ and $b \in \pi_1(Y)$.

Q: what is the relationship between the compositions ab and ba in $\pi_1(X \vee Y)$?

A: Nothing! We cannot switch the order of the two maps without collapsing one of them to the point, in which case it is the identity element.

To formalize this we introduce the free product.

2.1 Free product

Definition 2.1. Let $\{G_\alpha\}_{\alpha\in I}$ be groups. Define the *free product* $*_{\alpha\in I}G_\alpha$ as the group of formal words $g_1g_2\dots g_n$ with $g_i \in G_{\alpha_i}\setminus\{e\}$ and adjacent g_i, g_{i+1} are not from the same group $\alpha_i \neq \alpha_{i+1}$. Composition is given by concatenation $(g_1\dots g_n)(g'_1\dots g'_k) = g_1\dots g_n g'_1\dots g'_k$ where if g_n and g'_1 belong to the same group we multiply them. The unit is the empty word, and the inverse is $g_n^{-1}\dots g_1^{-1}$.

You will see later that $\pi_1(\bigvee_{\alpha\in I}X_\alpha) = *_{\alpha\in I}\pi_1(X_\alpha)$. For now, we will talk only about the algebraic side.

There are obvious homomorphisms $i_\alpha\colon G_\alpha \to *_{\alpha\in I}G_\alpha$ given by the length 1 words (or 0 for the unit).

Proposition 2.2. *$*_{\alpha\in I}G_\alpha$ is the coproduct of G_α in* Grp

Proof. Given homomorphisms $f_\alpha\colon G_\alpha \to H$ we can define $f\colon *_{\alpha\in I}G_\alpha \to K$ by

$$f(g_1g_2\dots g_n) = f_{\alpha_1}(g_1)f_{\alpha_2}(g_2)\dots f_{\alpha_n}(g_n)$$

where $g_i \in G_{\alpha_i}$. This homomorphism satisfies $f\circ i_\alpha = f_\alpha$, and if another $\tilde{f}$ satisfies $\tilde{f}\circ i_\alpha = f_\alpha$, then

$$\tilde{f}(g_1\dots g_n) = \tilde{f}(g_1)\dots\tilde{f}(g_n) = f_{\alpha_1}(g_1)\dots f_{\alpha_n}(g_n) = f(g_1\dots g_n)$$

so f is unique. □

Definition 2.3. The free group on a set S is $\langle S\rangle := *_{s\in S}\mathbb{Z}\langle s_i\rangle$. Explicitly, elements of $\langle S\rangle$ are formal words $s_1^{k_1}s_2^{k_2}\dots s_n^{k_n}$ for $s_i\in S$, $k_i\in\mathbb{Z}$. The free group extends to a functor $\langle -\rangle\colon \mathrm{Set}\to\mathrm{Grp}$.

Proposition 2.4. *$\langle -\rangle\colon \mathrm{Set}\to\mathrm{Grp}$ is left adjoint to the forgetful functor $U:\mathrm{Grp}\to\mathrm{Set}$, meaning there is a (natural) bijection $\hom_{\mathrm{Set}}(S, U(G)) \simeq \hom_{\mathrm{Grp}}(\langle S\rangle, G)$.*

Proof. A map of set $S\to U(G)$ corresponds to choosing elements $g_s\in G$ for every $s\in S$. Choosing an element corresponds to a homomorphism $f_s\colon\mathbb{Z}\to G$. By the universal property of coproducts, such a collection of homomorphisms corresponds to a single homomorphism $f\colon *_{s\in S}\mathbb{Z}\to G$. You can fill in the details of naturality. □

Every group can be represented as a quotient of a free group; this is the "generators and relations" representation $\langle S \mid R\rangle = \langle S\rangle/N(R)$ for $R\subseteq\langle S\rangle$. We will later use this representation to show that every group is the π_1 of some space.

The next example is of a free product which is not a free group.

Example 2.5. Consider the projective special linear group $\mathrm{PSL}(2,\mathbb{Z}) := \mathrm{SL}(2,\mathbb{Z})/\{I,-I\}$. Using row reductions and the Euclidean algorithm, it can be shown that $\mathrm{PSL}(2,\mathbb{Z})$ is generated by two elements

$$\alpha = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} \qquad \beta = \begin{pmatrix}1 & -1\\ 1 & 0\end{pmatrix}$$

Consider the inclusion of cyclic subgroups $\langle\alpha\rangle, \langle\beta\rangle \hookrightarrow \mathrm{PSL}(2,\mathbb{Z})$, which are of order 2 and 3 respectively. By the universal property, we get a homomorphism $\langle\alpha\rangle * \langle\beta\rangle \to \mathrm{PSL}(2,\mathbb{Z})$, we want to prove that this map is an isomorphism, implying $\mathrm{PSL}(2,\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$. Elements of the free product $w \in \langle\alpha\rangle * \langle\beta\rangle$ are words of the form $w = g_1 \dots g_n$, where g_i alternates between α and $\beta^{\pm 1}$, and the above map sends w to the corresponding multiplication in $\mathrm{PSL}(2,\mathbb{Z})$. The fact the α, β generate $\mathrm{PSL}(2,\mathbb{Z})$ precisely tells us that this map is surjective. To show that this map is injective, we need to show that the only word with trivial multiplication is the trivial word.

$\mathrm{PSL}(2,\mathbb{Z})$ is also called the modular group, due to its action on the upper half complex plain by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.z = \frac{az+b}{cz+d}$$

In particular, $\mathrm{PSL}(2,\mathbb{Z})$ acts on $\mathbb{R}$. Our two generators act as

$$\alpha.z = \frac{-1}{z}$$

$$\beta.z = \frac{z-1}{z} = 1 - \frac{1}{z}$$

$$\beta^{-1}.z = \frac{1}{1-z}$$

Notice that $\alpha(\mathbb{R}_{>0}) \subseteq \mathbb{R}_{<0}$ and $\beta^{\pm 1}(\mathbb{R}_{<0}) \subseteq \mathbb{R}_{>0}$ Suppose there exists a non-trivial word $w = g_1 \dots g_n \in \langle\alpha\rangle * \langle\beta\rangle$ which has a trivial multiplication in $\mathrm{PSL}(2,\mathbb{Z})$. In particular $w(\mathbb{R}_{>0}) = \mathbb{R}_{>0}$ and $w(\mathbb{R}_{<0}) = \mathbb{R}_{<0}$, so n must be even. We may assume that w ends with α, conjugating by α if necessary, so $w = \beta^{\pm 1} \dots \alpha$. Now if $w = \beta \dots \alpha$, then $w(\mathbb{R}_{>0}) \subseteq \beta(\mathbb{R}_{<0}) \subseteq \mathbb{R}_{>1}$, and if $w = \beta^{-1} \dots \alpha$, then $w(\mathbb{R}_{>0}) \subseteq \beta^{-1}(\mathbb{R}_{<0}) \subseteq \mathbb{R}_{<1}$. In particular, w can't be the identity on $\mathbb{R}_{<0}$.