Algebraic topology - Recitation 8

December 23, 2024

1 Comments on homework

1.1 CW-complexes

First, a general note on the definition of CW-complexes. We defined a CW-structure as a filtration $\varnothing = X^{-1} \subseteq X^0 \subseteq \cdots \subseteq X$ with cells $\Phi^n_\alpha \colon D^n \to X^n$ whose boundaries land in X^{n-1}, such that X^n is built from X^{n-1} by gluing the n-cells along their boundary. It is good to reiterate the difference between the *data* of a CW-complex and the *conditions* this data satisfies. The filtration is not a crucial part of the data, as it can be reconstructed from the cells $\Phi^n_\alpha \colon D^n \to X$. To reconstruct the filtration fromn this data, start with X^0, which only depends on the number of 0-cells, and iteratively build X^n from X^{n-1} by gluing the n-cells along their boundary. In fact, we see that we don't even need the full n-cells, but only their boundary maps $\varphi^n_\alpha \colon S^{n-1} \to X$.

However, if we are given only the data of the n-cells, we still need to check that it satisfies the conditions of a CW-complex. To phrase this condition for n-cells, we need to first define X^{n-1}, which is defined inductively if we checked the condition for lower cells. For that reason, it is simpler to give the filtration as part of the definition.

Suppose $Y \subseteq X$ is a subcomlex. Visually, X is built from cells of various dimensions, and some of them belong to Y. In X/Y, all the cells that came from Y are replaced by a single point. Thus, X/Y has all the cells in X which are not in Y together with a new 0-cell $[Y] \in X/Y$. For $\Phi^n_\alpha \colon D^n \to X$ a cell that does not belong to Y, the corresponding cell in X/Y will be $q \circ \Phi^n_\alpha$. Note that at every filtration level X^n, the cells coming from Y that appear in X^n are those of dimension $\leq n$, so $X^n \cap Y = Y_n$. The corresponding filtration is then $(X/Y)^n = X^n/Y^n$, and the boundary of $q \circ \Phi^n_\alpha$ indeed lands in X^{n-1}/Y^{n-1}.

A common mistake was taking all the cells (including those from Y) and replacing them with $q \circ \Phi^n_\alpha$. To see how this fails, consider $X/X = \mathrm{pt}$. Even if X had higher cells, X/X cannot have any cell other than a single 0-cell. Generally, the gluing should not identify any point in the interior of a cell, and if Φ^n_α is a cell from Y, then $q \circ \Phi^n_\alpha$ identifies all of it's point to a single point.

Let us see why the gluing condition is satisfied. For $n = 0$, it is enough to count and see that we have $|I_0| - |J_0| + 1$ 0-cells. A map $X^n/Y^n \to Z$ is equivalent to a map $X^n \to Z$ that is constant on Y^n. Giving a map from $X^n \to Z$ is equivalent to giving maps from X^{n-1} and from $\bigsqcup_{\alpha \in I_n} D^n$ that agree on the boundary, and saying that the map is constant on Y is equivalent to factoring through

X^{n-1}/Y^{n-1} and only including those n-cells which are not in Y. Thus, we get the pushout square

$$\begin{array}{ccc} \coprod_{\alpha\in I_n\setminus J_n} S^{n-1}_\alpha & \longrightarrow & \coprod_{\alpha\in I_n\setminus J_n} D^n_\alpha \\ \downarrow & & \downarrow \\ X^{n-1}/Y^{n-1} & \longrightarrow & X^n/Y^n. \end{array}$$

It is important that we removed the n-cells coming from Y – else we could have a non-constant map $D^n \to Z$ from a cell belonging to Y, which could not come from a map $X^n/Y^n \to Z$.

1.2 Inclusions and retract

An inclusion of subspaces does not generally induce an injection on homology, otherwise life would be simple and boring. For example, the inclusion $S^{n-1} \hookrightarrow D^n$ induces $\mathbb{Z} \to 0$ on $n-1$ homology. However, every functor, including homology, preserves retractions. That is, if we have $Y \overset{i}{\hookrightarrow} X \overset{r}{\to} Y$ such that $r \circ i = \mathrm{id}_Y$, then $\mathrm{H}_n(Y) \xrightarrow{i_*} \mathrm{H}_n(X) \xrightarrow{r_*} \mathrm{H}_n(Y)$ also satisfies $r_* \circ i_* = \mathrm{id}_{\mathrm{H}_n(Y)}$. In particular, we know that functions that have a retract are injective, so in this case i_* is indeed injective. Beware – even though in Set the converse is true, every injective function has a retract, this is not true in Grp or Ab. For example, $\mathbb{Z}/2\mathbb{Z} \xhookrightarrow{\times 2} \mathbb{Z}/4\mathbb{Z}$ or $\mathbb{Z} \hookrightarrow \mathbb{Q}$ do not have retracts. Such a retract exists precisely when the short exact sequence $0 \to A \hookrightarrow B \to B/A \to 0$ splits, giving $B \simeq A \oplus B/A$.

2 Calculating the fundamental group

We calculated the fundamental group of basic spaces:

(1) $\pi_1(\mathrm{pt}) = 0$

(2) $\pi_1(S^1) \simeq \mathrm{H}_1(S^1) \simeq \mathbb{Z}$ (you will prove in exercise)

(3) $\pi_1(S^n) = 0$ for $n > 0$.

We also know how to calculate the fundamental group of several constructions:

(1) $\pi_1(X \times Y) \simeq \pi_1(X) \times \pi_1(Y)$ (in exercise).

(2) (Van-Kampen) If $X = U \cup V$ is an open covering such that $U \cap V$ is path connected and $x_0 \in U \cap V$, the inclusions induce maps on π_1:

$$\begin{array}{ccc} \pi_1(U\cap V) & \longrightarrow & \pi_1(U) \\ \downarrow & & \downarrow \\ \pi_1(V) & \longrightarrow & \pi_1(X) \end{array}$$

which is a pushout. In the exercise you were asked to give a formula for pushouts in groups. We will present the formula (you still need to prove it is the pushout!): Given $\phi\colon G \to H$ and

$\psi\colon G \to K$, the pushout

$$\begin{array}{ccc} G & \xrightarrow{\phi} & H \\ {\scriptstyle\psi}\downarrow & \ulcorner & \downarrow \\ K & \longrightarrow & H *_G K \end{array}$$

is given by the *amalgamated free product* $H *_G K = H * K / N(\{\phi(g)\psi(g)^{-1} \mid g \in G\})$.

(3) In particular, if X, Y are pointed with a contractible neighborhood of the base point, then $\pi_1(X \vee Y) = \pi_1(X) * \pi_1(Y)$.

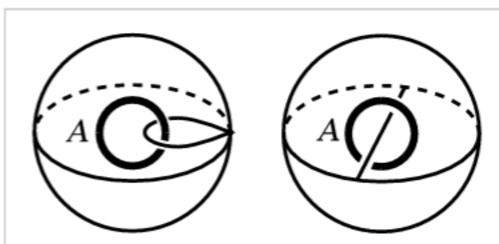
We will use those tools to calculate the fundamental group of more complicated spaces.

Example 2.1. Consider the torus $T = S^1 \times S^1$, then $\pi_1(T) = \mathbb{Z}^2$.

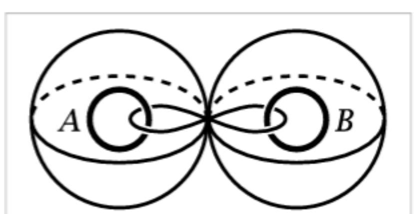
Example 2.2. $\pi_1(S^1 \vee S^1) \simeq \mathbb{Z} * \mathbb{Z} = F_2$ the free group on two generators. Generally, $\pi_1(\bigvee_{i=1}^n S^1) \simeq F_n$, the free group on n generators.

Example 2.3. A *knot* is an embedding $S^1 \hookrightarrow \mathbb{R}^3$. A *link* is an embedding $\bigsqcup S^1 \hookrightarrow \mathbb{R}^3$. We identify knots and links $f, g\colon \bigsqcup S^1 \hookrightarrow \mathbb{R}^3$ up to an ambient isotopy, which is a homotopy $h_t\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that $h_0 = \mathrm{id}_{\mathbb{R}^3}$, h_t is a homeomorphism and $h_1 \circ f = g$. Consider the links $L \subseteq \mathbb{R}^3$ of two circles linked, and $U \subseteq \mathbb{R}^3$ two circles unlinked. One way to show that those links are different, is to show that their complements $\mathbb{R}^3 - L$ and $\mathbb{R}^3 - U$ have a different fundamental group.

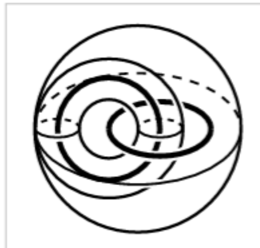
Let us start with the simpler example of the unknot S^1. $\mathbb{R}^3 - S^1$ has a deformation retract to $S^1 \vee S^2$



Similarly, $\mathbb{R}^3 - U$ deformation retracts to $S^1 \vee S^1 \vee S^2 \vee S^2$



thus $\pi_1(\mathbb{R}^3 - U) \simeq F_2$. On the other hand, $\mathbb{R}^3 - L$ deformation retracts to $T \vee S^2$,



so $\pi_1(\mathbb{R}^3 - L) \simeq \mathbb{Z}^2$. Note that we could not differentiate the two links using homology, because.

$$\mathrm{H}_n(\mathbb{R}^3 - U) \simeq \mathrm{H}_n(\mathbb{R}^3 - L) \simeq \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}^2 & n = 1, 2 \\ 0 & n > 2 \end{cases}$$

2.1 CW-complexes

We will start by presenting a different approach to calculating $\pi_1(T)$.

Example 2.4. Consider a covering $T = U \cup V$ where U is an open disk around some $x_0 \neq x \in T$ which contains x_0, and $V = T - \{x\}$. Up to a deformation retract, $U \simeq \mathrm{pt}$, $V \simeq S^1 \vee S^1$ and $U \cap V \simeq S^1$. By the previous example, $\pi_1(V) \simeq F_2$ and $\pi_1(U \cap V) \simeq \mathbb{Z}$, and the inclusion $U \cap V \hookrightarrow V$ induces a homomorphism $\mathbb{Z} \to F_2$ which chooses $aba^{-1}b^{-1}$. By Van-Kampen,

$$\pi_1(T) \simeq \pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V) \simeq 0 *_{\mathbb{Z}} F_2 = F_2/N(\langle aba^{-1}b^{-1} \rangle) = F_2^{\mathrm{ab}} \simeq \mathbb{Z}^2$$

In this example we used the CW-structure of T whose 1-skeleton is $S^1_a \vee S^1_b$, and a 2-cell is attached along $ab\bar{a}\bar{b}$, and this 2-cell gave us a null-homotopy of its boundary. This is true in general.

Proposition 2.5. *Let X be a path-connected pointed space, and let $\varphi\colon S^1 \to X$ be some pointed map. Consider the space $Y = D^2 \cup_{S^1, \varphi} X$, a gluing of a 2-cell to X along φ. Then $\pi_1(Y) \simeq \pi_1(X)/N([\varphi])$.*

Proof. Let $U = Y - X$ and $V = Y - \{0\}$. U is contractible, V deformation retracts to X, and $U \cap V$ is path-connected and deformation retracts to S^1. Note that the basepoint x_0 does not belong to $U \cap V$, but we can move the basepoint to some $x_1 \in U \cap V$, and consider a path λ from x_0 to x_1 going through D^2. The inclusion $U \cap V \hookrightarrow V$ induces a homomorphism $\mathbb{Z} \simeq \pi_1(U \cap V, x_1) \to \pi_1(V, x_1)$ which chooses $[\bar{\lambda}\varphi\lambda] \in \pi_1(V, x_1)$. Thus, by Van-Kampen

$$\pi_1(Y, x_1) \simeq 0 *_{\mathbb{Z}} \pi_1(V, x_1) = \pi_1(V, x_1)/N([\bar{\lambda}\varphi\lambda]).$$

The choice of λ induces an isomorphism $\pi_1(Y, x_1) \simeq \pi_1(Y, x_0)$ which sends $[\alpha]$ to $[\lambda\alpha\bar{\lambda}]$, so $[\bar{\lambda}\varphi\lambda]$ is sent to $[\varphi]$. Under this isomorphism, wee get

$$\pi_1(Y, x_0) \simeq \pi_1(V, x_0)/N([\varphi]) \simeq \pi_1(X, x_0)/N([\varphi])$$

□

Example 2.6. Consider the CW-structure on $\mathbb{RP}^2$ whose 1-skeleton is S^1 and the attaching map of the 2-cell is $S^1 \xrightarrow{(-)^2} S^1$. This map is of degree 2, so it corresponds to the element $2 \in \pi_1(S^1) \simeq \mathbb{Z}$. We get that

$$\pi_1(\mathbb{RP}^2) = \mathbb{Z}/N(2) = \mathbb{Z}/2\mathbb{Z}$$

Now if $\varphi\colon S^1 \to X$ is an unpointed map, i.e. the basepoint $s_0 \in S^1$ is not sent to $x_0 \in X$, we can choose a path γ from x_0 to $\varphi(s^0)$ and use the isomorphism $\pi_1(X, \varphi(s_0)) \simeq \pi_1(X, x_0)$ which sends $[\varphi] \in \pi_1(X, \varphi(s_0))$ to $[\gamma\varphi\bar{\gamma}] \in \pi_1(X, x_0)$. Under this isomorphism, we have

$$\pi_1(Y, x_0) \simeq \pi_1(X, x_0)/N([\gamma\varphi\bar{\gamma}])$$

Generally, if Y is formed by gluing (finitely many) 2-cells with attaching maps $\varphi_\alpha\colon S^1 \to X$, and γ_α are paths from x_0 to $\phi_\alpha(s_0)$, then

$$\pi_1(Y, x_0) = \pi_1(X, x_0)/N(\langle[\gamma_\alpha\varphi_\alpha\bar{\gamma}_\alpha]\rangle)$$

What happens when we glue a higher dimensional cell?

Proposition 2.7. *Let X be a path-connected space, and let $\varphi\colon S^{n-1} \to X$ be some map for $n > 2$. Consider the space $Y = D^n \cup_{S^{n-1},\varphi} X$ a gluing of an n-cell to X along φ. Then $\pi_1(Y) \simeq \pi_1(X)$.*

Proof. As before, choose $U = Y - X$ and $V = Y - 0$. U is contractible, V deformation retracts to X and $U \cap V$ deformation retracts to S^{n-1}. By Van-Kampen,

$$\pi_1(Y) \simeq \pi_1(D^n) *_{\pi_1(S^{n-1})} \pi_1(X) \simeq 0 *_0 X = X.$$

(a similar trick as above can be used to fix the basepoint). ☐

Corollary 2.8. *Given a (finite) CW-complex X, the 2-skeleton inclusion $X^2 \hookrightarrow X$ induces an isomorphism on π_1.*