Algebraic topology - Recitation 9

December 30, 2024

1 Equivalence of categories

Any meaningful mathematical notion is isomorphism invariant. In particular, we usually don't want to assert that two constructions are equal, only isomorphic. In the category of categories, we have the following notion of isomorphism:

Definition 1.1. An functor $F\colon \mathscr{C} \to \mathscr{D}$ is an *isomorphism* if there exist a functor $G\colon \mathscr{D} \to \mathscr{C}$ such that

$$G \circ F = \mathrm{id}_{\mathscr{C}} \qquad\qquad F \circ G = \mathrm{id}_{\mathscr{D}}$$

The assertion $G \circ F = \mathrm{id}_{\mathscr{C}}$ means that for every $X \in \mathscr{C}$ we have an equality $G(F(X)) = X$. This breaks the core tenet – we assert that two objects in $\mathscr{C}$ are equal, instead of demanding an isomorphism. Recall that functors $\mathrm{Fun}(\mathscr{C}, \mathscr{D})$ form a category, where the isomorphisms are natural isomorphisms. The above definition, of F being an isomorphism of categories, is not invariant under natural isomorphisms. Instead, the better way to identify between categories is with equivalence of categories:

Definition 1.2. An functor $F\colon \mathscr{C} \to \mathscr{D}$ is an *equivalence of categories* if there exist a functor $G\colon \mathscr{D} \to \mathscr{C}$ and natural isomorphisms

$$G \circ F \simeq \mathrm{id}_{\mathscr{C}} \qquad\qquad F \circ G \simeq \mathrm{id}_{\mathscr{D}}$$

If F is an equivalence of categories, then the inverse G is unique (up to natural isomorphism).

Example 1.3. Consider $\mathscr{C} = \{1\}$ the point, and $\mathscr{D}$ the category with two points and an isomorphism between them $f\colon 1 \xrightarrow{\sim} 2$.

$$\begin{tikzcd}
 & & 2 \arrow[d, bend left, "f^{-1}"] \\
1 \arrow[rr, dashed, mapsto, shift left, "F"] & & 1 \arrow[u, bend left, "f"] \arrow[ll, dashed, mapsto, shift left, "G"]
\end{tikzcd}$$

Those categories are not isomorphic, as they have a different number of objects. Define $F\colon \mathscr{C} \to \mathscr{D}$ by $1 \mapsto 1$ and $G\colon \mathscr{C} \to \mathscr{D}$ by $1, 2 \mapsto 1$. We have $GF(1) = 1$, so $GF = \mathrm{id}_{\mathscr{C}}$. On the other hand, $FG(1) = FG(2) = 1$, so $FG \neq \mathrm{id}_{\mathscr{D}}$. However, there is a natural isomorphism $\alpha\colon FG \xrightarrow{\sim} \mathrm{id}_{\mathscr{D}}$, given by $\alpha_1 = \mathrm{id}_1\colon 1 \xrightarrow{\sim} 1$ and $\alpha_2 = f\colon 1 \xrightarrow{\sim} 2$. Thus, $\mathscr{C}$ and $\mathscr{D}$ are equivalent categories.

1.1 Fundamental groupoids

The definition of equivalence of categories might seem similar to homotopy equivalence, where instead of homotopies of maps we have natural isomorphisms of functors. There is indeed a connection between the two concepts. Recall that for a space X, the fundamental groupoid $\pi_{\leq 1}(X)$ is the category whose objects are points in X and whose morphisms are paths up to homotopy.

Proposition 1.4. *Let $f, g\colon X \to Y$ be homotopic maps, then $f_*, g\colon \pi_{\leq 1}(X) \to \pi_{\leq 1}(Y)$ are naturally isomorphic functors.*

Proof. Let $h\colon X \times I \to Y$ be a homotopy between f and g. For every $x \in X$, $h(x, -)\colon I \to Y$ defines a path in Y between $f(x)$ and $g(x)$. Such a path corresponds to an isomorphism in the fundamental groupoid $\alpha_x\colon f_*(x) \xrightarrow{\sim} g_*(x) \in \pi_{\leq 1}(Y)$. We want to show that α assembles into a natural isomorphism, meaning that for every path $p\colon x \to y \in \pi_{\leq 1}(X)$ the following square commutes:

$$\begin{tikzcd} f_*(x) \arrow[r, "\alpha_x"] \arrow[d, "f_*(p)"'] & g_*(x) \arrow[d, "g_*(p)"] \\ f_*(y) \arrow[r, "\alpha_y"] & g_*(y) \end{tikzcd}$$

Consider $H\colon I^2 \to Y$ given by $H(s,t) = h(p(s), t)$. The edges of I^2 are mapped to

$$\begin{aligned} H(0,-) = h(x,-) = \alpha_x \quad & H(1,-) = h(y,-) = \alpha_y \\ H(-,0) = h(p(-),0) = f_*(p) \quad & H(-,1) = h(p(-),1) = g_*(p) \end{aligned}$$

So, moving through the interior of I^2, we get a homotopy between the composition of paths above. □

Corollary 1.5. *If $f\colon X \to Y$ is a homotopy equivalence, then $f_*\colon \pi_{\leq 1}(X) \to \pi_{\leq 1}(Y)$ is an equivalence of categories.*

Proof. There exists $g\colon Y \to X$ and homotopies $gf \sim \mathrm{id}_X$ and $fg \sim \mathrm{id}_Y$. By the above proposition, we get natural isomorphisms $g_* f_* \simeq \mathrm{id}_{\pi_{\leq 1}(X)}$ and $f_* g_* \simeq \mathrm{id}_{\pi_{\leq 1}(Y)}$. □

1.2 Grothendick construction for sets

For a set B, define $\mathrm{Set}_{/B}$ the category of sets over B, whose:

- objects are sets A with a function $A \xrightarrow{f} B$
- morphisms are functions $g\colon B \to B'$ such that the triangle commutes

$$\begin{tikzcd} A \arrow[rr, "g"] \arrow[dr, "f"'] & & A' \arrow[dl, "f'"] \\ & B & \end{tikzcd}$$

On the other hand, we can also consider B as a discrete category, in which case $\mathrm{Fun}(B, \mathrm{Set})$ is the category of B-indexed families of sets.

Definition 1.6. Define the following functors:

(1) The fiber $\mathrm{fib}\colon \mathrm{Set}_{/B} \to \mathrm{Fun}(B, \mathrm{Set})$: for $A \xrightarrow{f} B \in \mathrm{Set}_{/B}$, define $\mathrm{fib}(f) \in \mathrm{Fun}(B, \mathrm{Set})$ by taking the fibers $\mathrm{fib}(f)_b = f^{-1}(b)$. Given $A, A' \in \mathrm{Set}_{/B}$ and $g\colon A \to A'$ such that $f = f' \circ g$, note that

$$a \in \mathrm{fib}(f)_b \iff f(a) = b \iff f'(g(a)) = b \iff g(a) \in \mathrm{fib}(f')_b$$

Thus, g induces a map $\mathrm{fib}(f) \to \mathrm{fib}(f')$ by acting on each fiber separately.

(2) The Grothendick construction $\int\colon \mathrm{Fun}(B, \mathrm{Set}) \to \mathrm{Set}_{/B}$: for a family of sets $(A_b)_{b\in B}$, define $\int A_b \in \mathrm{Set}_{/B}$ as the set $\bigsqcup_{b\in B} A_b$ with the map $\pi\colon \bigsqcup_{b\in B} A_b \to B$ sending the component A_b to b. For a family of functions $g_b\colon A_b \to A'_b$, there is an induced map $g\colon \sqcup A_b \to \sqcup A'_b$ acting on each fiber separately. For concreteness, let us choose a model for the disjoint union, say $\bigsqcup_{b\in B} A_b = \bigcup_{b\in B} A_b \times \{b\}$.

Proposition 1.7. *There is an equivalence of categories* $\mathrm{Set}_{/B} \simeq \mathrm{Fun}(B, \mathrm{Set})$ *given by* fib *and* $\int$.

Proof. Let $(A_b)_{b\in B} \in \mathrm{Fun}(B, \mathrm{Set})$. Note that $\mathrm{fib}(\pi)_b = \pi^{-1}(b) = A_b \times \{b\} \simeq A_b$, and this isomorphism is natural,

$$\begin{tikzcd} A_b \times \{b\} \arrow[r, "\sim"] \arrow[d] & A_b \arrow[d] \\ A'_b \times \{b\} \arrow[r, "\sim"] & A'_b \end{tikzcd}$$

so we get a natural isomorphism $\mathrm{fib} \circ \int \xrightarrow{\sim} \mathrm{id}_{Set_{/B}}$.

On the other hand, suppose we have $A \xrightarrow{f} B \in \mathrm{Set}_{/B}$, and denote $A_b = \mathrm{fib}(f)_b = f^{-1}(b)$. The inclusions $A_b \hookrightarrow A$ induce a map $\bigsqcup A_b \to A$. This map is injective, because all fibers are disjoint, and surjective, because every element belongs to some fiber, so it is an isomorphism of sets. Moreover, the triangle commutes

$$\begin{tikzcd} \bigsqcup A_b \arrow[rr, "\sim"] \arrow[dr, "\pi"'] & & A \arrow[dl, "f"] \\ & B & \end{tikzcd}$$

Naturality then boils down to the statement that applying a function on A is the same as applying the function on each fiber seperately. $\square$

Example 1.8. (1) $\mathrm{Set}_{/\mathrm{pt}} \simeq \mathrm{Fun}(\mathrm{pt}, \mathrm{Set}) \simeq \mathrm{Set}$

(2) $\mathrm{Set}_{/\{0,1\}} \simeq \mathrm{Fun}(\{0,1\}, \mathrm{Set})$: a set with a map $A \to \{0,1\}$ has the same data as considering the two fibers A_0, A_1. Note that, if we fix A, choosing a map $A \to \{0,1\}$ is equivalent to choosing a subset $A_0 \subseteq A$ (the subset A_1 is then the complement). For that reason. $\{0,1\}$ is called the *subobject classifier*.

2 Colimits

Every meaningful categorical construction is invariant under equivalence of categories. In particular, this is true for colimits, as you will see in the exercise. First, let us give the (long overdue) definition of a general colimit, starting with the simplest colimit, the initial object.

Definition 2.1. An object $X \in \mathscr{C}$ is called initial if for every $Y \in \mathscr{C}$ there is a unique map $X \to Y$.

Lemma 2.2. *The initial object, if it exists, is unique up to a unique isomorphism.*

Proof. Suppose $X, X' \in \mathscr{C}$ are initial, there are unique map $X \to X'$ and $X' \to X$, and the compositions

$$X \to X' \to X$$

$$X' \to X \to X'$$

must be the unique maps $X \to X$ and $X' \to X'$ respectively, so they must be equal id_X and $\mathrm{id}_{X'}$ respectively. □

Example 2.3. The empty set $\varnothing \in \mathrm{Set}$ is the initial set. Similarly, $\varnothing \in \mathrm{Top}$ and $\varnothing \in \mathrm{Cat}$ are initial. The trivial group 0 is initial in Grp and Ab.

Let I be some category, thought of as an indexing category, and let $D\colon I \to \mathscr{C}$ be a functor, thought of as an I-shaped diagram in $\mathscr{C}$. A cocone in $\mathscr{C}$ under D is an object $X \in \mathscr{C}$ with maps from $D(i)$ making the extended diagram commute. Formally, we define the following category:

Definition 2.4. The *category of cocones* in $\mathscr{C}$ under D, denoted $\mathscr{C}_{D/}$, has:

- objects $X \in \mathscr{C}$ together with maps $f_i\colon D(i) \to X$ such that for every $\sigma\colon i \to j$, $f_i = f_j \circ D(\sigma)$.
- morhpisms $(X, f_i) \to (X', f_i')$ are maps $g\colon X \to X'$ such that $g \circ f_i = f_i'$.

The *colimit* of D is the initial object of $\mathscr{C}_{D/}$, if it exists, and is denoted $\operatorname{colim} D$.

Example 2.5. Consider $I = \varnothing$, in which case there is only the empty diagram $D\colon \varnothing \to \mathscr{C}$. The category $\mathscr{C}_{D/}$ is then just an object $X \in \mathscr{C}$ with no extra maps, so $\mathscr{C}_{D/} \simeq \mathscr{C}$ and the colimit of D is the initial object of $\mathscr{C}$.

Example 2.6. We have seen examples of colimits for the following shapes of diagrams:

(1) Coproduct: $I = \bullet\ \bullet$.

(2) Pushout: $I = \bullet \leftarrow \bullet \to \bullet$.

(3) Sequential colimit: $I = \bullet \to \bullet \to \dots$

Remark 2.7. There is a dual construction of a limit: the limit of $D\colon I \to \mathscr{C}$ is the colimit of $D^{\mathrm{op}}\colon I^{\mathrm{op}} \to \mathscr{C}^{\mathrm{op}}$. More explicitly, an object $X \in \mathscr{C}$ is terminal if it has a unique map from every $Y \in \mathscr{C}$, and a cone $(X, f_i) \in \mathscr{C}_{/D}$ is an object $X \in \mathscr{C}$ with maps $f_i\colon X \to D(i)$ making the diagram commute. The limit is then a terminal cone.