Algebraic topology - Recitation 10

January 6, 2025

1 On Homework

Remark 1.1. From now on, the normal closure will be denoted $\langle\langle R\rangle\rangle$ while the normalizer will be denoted $N(H)$.

1.1 The $k \times k$ lemma

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & & & 0 & & \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \\
0 & \longleftarrow & A_{0,k-1} & \longleftarrow & A_{1,k-1} & \longleftarrow & \cdots & \longleftarrow & A_{k-1,k-1} & \longleftarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \\
 & & \vdots & & \vdots & & & & \vdots & & \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \\
0 & \longleftarrow & A_{0,1} & \longleftarrow & A_{1,1} & \longleftarrow & \cdots & \longleftarrow & A_{k-1,1} & \longleftarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \\
0 & \longleftarrow & A_{0,0} & \longleftarrow & A_{1,0} & \longleftarrow & \cdots & \longleftarrow & A_{k-1,0} & \longleftarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \\
 & & 0 & & 0 & & & & 0 & &
\end{array}$$

Suppose all rows and columns except the first are exact. First of all, it follows that the first row and column also satisfy the chain rule. Without loss of generality we will prove for the row, Let $a \in A_{n+2,0}$, we want to prove that $d_<^2(a) = 0$. By surjectivity, there is some $b \in A_{n+2,1}$ such that $a = d_\vee(b)$, and

$$d_<^2(a) = d_<^2 d_\vee(b) = d_\vee d_<^2(b) = 0$$

We can change every even vertical map to be negative for the bicomplex, this doesn't affect homology. The E^1 pages are the horizontal/vertical homologies, which are zero except for the first line $\mathrm{H}_n(A_{\bullet,0}), \mathrm{H}_n(A_{0,\bullet})$. The spectral sequence collapses at this point, so this is also the E^∞ page. Beware! Apriori, there is no reason for $E^{<\infty}$ and $E^{\vee\infty}$ to be the same; they are both associated graded on different filtrations of $\mathrm{H}_n(\mathrm{Tot}(A))$. However, when the associated graded has a single non-zero term

$$\dots, 0, 0, G, 0, 0 \dots$$

then the filtration is constant except at this spot, where is must change from 0 to G

$$\dots \subseteq 0 \subseteq 0 \subseteq G \subseteq G \subseteq G \subseteq \dots$$

and in this case G is the total group. So for those specific E^∞ pages, we get

$$\mathrm{H}_n(A_{\bullet,0}) = \mathrm{H}_n(\mathrm{Tot}(A)) = \mathrm{H}_n(A_{0,\bullet}).$$

1.2 Free product of presentations

Recall the $\langle S \mid R\rangle = \langle S\rangle/\langle\langle R\rangle\rangle$. We want to prove that $\langle S_1 \mid R_1\rangle * \langle S_2 \mid R_2\rangle = \langle S_1 \sqcup S_2 \mid R_1 \sqcup R_2\rangle$. Dealing with the explicit structure of free groups and normal closures is hard. The easiest proof is to show that they both have the same universal property. First, we want to describe the universal property of quotient by the normal closure.

Lemma 1.2. *Let $R \subseteq G$ be a subgroup, then homomorphisms $\bar{f}\colon G/\langle\langle R\rangle\rangle \to H$ are the same as homomorphisms $f\colon G \to H$ such that $f(R) = 0$.*

Proof. The universal propertey of the quotient tells us that that homomorphisms $\bar{f}\colon G/\langle\langle R\rangle\rangle \to H$ are the same as homomorphisms $f\colon G \to H$ such that $\langle\langle R\rangle\rangle \subseteq \ker(f)$. But $\ker(f)$ is normal, so this is equivalent to $R \subseteq \ker(f)$. □

We will also use the fact that $\langle S_1 \sqcup S_2\rangle \simeq \langle S_1\rangle * \langle S_2\rangle$. This follows either by definition $\langle S\rangle = *_{s\in S}\mathbb{Z}$, or from the universal property.

Now notice that homomorphisms $\bar{f}\colon \langle S_1 \sqcup S_2 \mid R_1 \sqcup R_2\rangle \to H$ are the same as homomorphisms $f\colon \langle S_1 \sqcup S_2\rangle \to H$ such that $f(R_1 \sqcup R_2) = 0$ which are the same as $f_1\colon \langle S_1\rangle \to H$ and $f_2\colon \langle S_1\rangle \to H$ such that $f_1(R_1) = 0$ and $f_2(R_2) = 0$ which are the same as $\bar{f}_1\colon \langle S_1 \mid R_1\rangle \to H$ and $\bar{f}_2\colon \langle S_2 \mid R_2\rangle \to H$.

2 Galois correspondence of covering spaces

Let X be a path connected locally simply connected space, and denote $G = \pi_1(X)$. Today you proved that there is an equivalence of categories:

$$\{\text{covers of } X\} \simeq \{G\text{-sets}\}$$

which on pointed connected covers reduces to an equivalence

$$\{\text{Pointed connected covers of } X\} \simeq \{\text{Pointed transitive } G\text{-sets}\} \simeq \{\text{subgroups of } G\}$$

This equivalence is given by

$$(p\colon Y \to X) \mapsto (p_*(\pi_1(Y)) \subseteq G)$$
$$(H \subseteq G) \mapsto (\tilde{X}/H \to X)$$

where $\tilde{X} \to X$ is the universal cover. This correspondence has a striking resemblance to Galois correspondence of fields. This is not by accident: there is a way in algebraic geometry to realize field extensions as a form of covering spaces (étale morphism of schemes). We will prove statements about this correspondence that are analogs of statements in Galois theory of fields.

2.1 Deck transformations

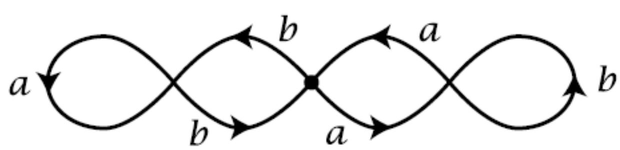
Let $Y \to X$ be a covering space. A *deck transformation* is an automorphism $Y \xrightarrow{\sim} Y$ of covering spaces. The group of deck transformation is denoted $\mathrm{Aut}_X(Y)$. For example, for the universal covering $\mathbb{R} \to S^1$, the deck transformation are integer shifts of $\mathbb{R}$.

If Y is path connected, the unique lifting property tells us that a deck transformation is determined by its value on a single point. Generally, a deck transformation is determined by its value on a point from every connected component.

Definition 2.1. A covering $p\colon Y \to X$ is called *normal* if for each $x \in X$ and every $y_1, y_2 \in p^{-1}(x_0)$ there exists a deck transformation $\tau\colon Y \to Y$ such that $\tau(y_1) = y_2$.

Let F denote the fiber of Y at some point, with its corresponding G-action. The equivalence of categories between covering space and G-sets gives us a bijection $\mathrm{Aut}_X(Y) \simeq \mathrm{Aut}_G(F)$, where $\mathrm{Aut}_G(F)$ is the automorphism group of F as a G-set. Under this bijection, a covering is normal if and only if the action of $\mathrm{Aut}_G(F)$ on F is transitive.

Example 2.2. The coverings $\mathbb{R} \to S^1$, as well as $(-)^n\colon S^1 \to S^1$, are normal. However, a covering like $\mathbb{R} \sqcup S^1 \to S^1$ is not normal, as there is no deck transformation between the components. For an example of a non-normal connected cover, consider the cover of $S^1 \vee S^1$ given by



The name "normal" is motivated by the following result:

Proposition 2.3. *Let $p\colon (Y, y_0) \to (X, x_0)$ be a (pointed) path-connected covering space, and let $H = p_*(\pi_1(Y)) \subseteq \pi_1(X) = G$. Then:*

(1) *The covering space is normal if and only if H is a normal subgroup of G.*

(2) $\mathrm{Aut}_X(Y) \simeq N(H)/H$*, where $N(H)$ is the normalizer of H in G.*

In particular, $\mathrm{Aut}_X(Y) \simeq G/H$ *when Y is normal, and for the universal cover we get* $\mathrm{Aut}_X(\tilde{X}) = \pi_1(X)$.

Using the equivalence of categories, we can reduce this statement to a purely algebraic one:

Lemma 2.4. *Let $H \subseteq G$ be a subgroup, and consider the G-set G/H. Then there is an isomorphism* $\mathrm{Aut}_G(G/H) \simeq N(H)/H$.

Proof. Suppose $n \in N(H)$, meaning $Hn = nH$. There is a right action of $N(H)$ on G/H given by $(kH).n = kHn = knH$. This right action respects the left action of G, $g.(kH).n = gknH$, so $(-).n$ is an automorphism of G-sets. Thus, this action defines a homomorphism $N(H) \to \mathrm{Aut}_G(G/H)$.

This homomorphism is surjective: Given $\tau \in \mathrm{Aut}_G(H)$, let $\tau(eH) = nH$. Because τ is an automorphism of G-sets, it follows that for every $h \in H$

$$nH = \tau(eH) = \tau(hH) = h\tau(eH) = hnH$$

meaning in particular that $hn \in nH$, so $Hn = nH$. Moreover, for every $kH \in G/H$ we have $\tau(kH) = k\tau(eH) = knH = (kH)n$, so τ is equal to the right action of n.

The kernel is H: It is immediate that for $h \in H$ the right action is trivial, $(kH)h = kHh = kH$. On the other hand, if $n \in N(H)$ has a trivial right action, then $Hn = H$, so in particular $n = en \in H$. □

Lemma 2.5. *A subgroup $H \subseteq G$ is normal if and only if the action of $\mathrm{Aut}_G(G/H)$ on G/H is transitive.*

Proof. Suppose H is normal. For every $g \in G = N(H)$, right multiplication by g defines an automorphism of G-sets, which satisfies $(eH)g = gH$. Now assume the action is transitive, so for every $g \in G$ there is some $n \in N(H)$ such that $nH = (eH)n = gH$. It follows that $g = nh$ for some $h \in H$, but $H \subseteq N(H)$ so $g \in N(H)$. □

Proposition. Follows from the above two lemmas under the identification $\mathrm{Aut}_X(Y) = \mathrm{Aut}_G(G/H)$, recalling that a covering is normal if the action of $\mathrm{Aut}_G(G/H)$ on G/H is transitive. □

2.2 Group actions

The group of deck transformations $\mathrm{Aut}_X(Y)$ acting on Y is a special case of a group acting on a space. Given a group G and a space Y, an *action* of G on Y is a homomorphism $G \to \mathrm{Aut}(Y)$ from G to the group of homeomorphism $Y \to Y$. We will consider specifically free actions, meaning that for every $g \neq e$ and every $y \in Y$, $gy \neq y$. In particular, a free action is faithful, so the homomorphism $G \to \mathrm{Aut}(Y)$ is injective. In fact, we would like a stronger, topological version of freeness:

(∗) Each $y \in Y$ has an open neighborhood U such that all images $g(U)$ are disjoint for different g. That is, if $g_1 \neq g_2$ then $g_1(U) \cap g_2(U) = \varnothing$, or equivalently if $g \neq e$ then $g(U) \cap U = \varnothing$.

The action of deck transformations on a connected covering space $p\colon Y \to X$, given by the inclusion $\mathrm{Aut}_X(Y) \hookrightarrow \mathrm{Aut}(Y)$ of covering-preserving automorphism into all automorphisms, satisfies (∗): Given $y \in Y$, suppose $x = p(y) \in X$. By definition of a covering space, there is a neighborhood $x \in V$ such that $p^{-1}(V) = \bigsqcup U_\alpha$, where $p|_{U_\alpha}\colon U_\alpha \to V$ is a homeomorphism. Suppose $y \in U_0$, every deck transformation τ sends U_0 to some U_α, and if $U_\alpha = U_0$ then it must send y to itself, implying $\tau = \mathrm{id}$.

For an example of a free action that does not satisfy (⋆), consider the action of $\mathbb{Z}$ on S^1 given by rotations by $2\pi\alpha$ where α is irrational.

Given an action of a G on Y, we can form a space Y/G, the quotient space under the equivalence relation of the orbits. This space is called the *orbit space*.

Proposition 2.6. *If an action of G on Y satisfies $(*)$, then:*

(1) *The quotient map $p\colon Y \to Y/G$ is a normal covering space.*

(2) *If Y is path-connected, then G is the group of deck transformation of $p\colon Y \to Y/G$.*

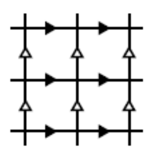
(3) *If Y is path-connected and locally simply connected, then G is isomorphic to $\pi_1(Y/G)/p_*(\pi_1(Y))$.*

Proof. Let $y \in Y$ and U as in $(*)$. The quotient map p identifies all disjoint neighborhood $g(U)$ homemorphicaly to $p(U)$, so we have a covering space. Each elemet of G acts as a deck transformation, so we have a homomorphism $G \to \mathrm{Aut}_{Y/G}(Y)$. Choosing some $y_0 \in Y$, we get a homomorphism in the other direction where a deck transformation τ is sent to the unique g such that $\tau(y_0) = gy_0$ The composition $G \to \mathrm{Aut}_{Y/G}(Y) \to G$ is clearly the identity, and if Y is path connected then $\mathrm{Aut}_{Y/G}(Y) \to G \to \mathrm{Aut}_{Y/G}(Y)$ is also the identity, because a deck transformation is determined by its action on a single point. The covering is normal, because for every $g_1y, g_2y \in p^{-1}([y])$, the deck transformation corresponding to $g_1g_2^{-1}$ sends one to the other. For the final statement, notice that if Y is locally path connected then Y/G is also locally path connected, so it follows from the above proposition. □

Suppose $p\colon Y \to X$ is a path connected, locally simply-connected normal cover, with $\pi_1(X) = G$ and $p_*\pi_1(Y) = N$. In particular $\mathrm{Aut}_X(Y) \simeq G/N$, and subgroups $H \subseteq \mathrm{Aut}_X(Y)$ correspond to intermediate $N \subseteq HN \subseteq G$. Such H corresponds to a factorization of p into two cover $Y \to Y/H \to X$, where $\pi_1(Y/H) = HN$.

Note that if Y is simply connected and G acts on Y satisfying $(*)$, then $G \simeq \pi_1(Y/G)$. This method gives us an alternative way to calculate $\pi_1(S^1) \simeq \pi_1(\mathbb{R}/\mathbb{Z}) \simeq \mathbb{Z}$, without using homology and Eckman-Hilton. It can be used to calculate the fundamental groups of more spaces.

Example 2.7. Consider $\mathbb{R}^2$ with the following grid:



Consider the action of $\mathbb{Z}^2$ on $\mathbb{R}^2$ preserving the grid, given by horizontal and vertical shifts. This action satisfies $(*)$, as every non identity element moves each point by at least 1 unit length. The orbit space of this action is $\mathbb{R}^2/\mathbb{Z}^2$, which is homeomorphic to the torus, so it follows that $\pi_1(\mathbb{T}) \simeq \mathbb{Z}^2$.