Algebraic topology - Recitation 11

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1 On Homework

Let (G, e) be a topological group, we want to prove that $\pi_1(G, e)$ is Abelian. The idea was to define an additional group structure on $\pi_1(G, e)$ by the multiplication of G, and use the Eckman-Hilton argument. Verifying that such a multiplication is homotopy invariant, and that it commutes with composition of paths, requires some routine homotopical arguments, which are very similar to things we've done before. In this case it is not too bad to run the arguments explicitly, but I want to present an alternative, which uses only previously proven claims and abstract nonsense.

Using the fact that π_1 commutes with products and that the multiplication of G is continuous, we get a homomorphism

$$\boxtimes\colon \pi_1(G) \times \pi_1(G) \simeq \pi_1(G \times G) \xrightarrow{m_*} \pi_1(G).$$

Automatically this map is homotopy invariant, and the fact that it is a homomorphism implies that

$$\boxtimes(a * b, c * d) = \boxtimes((a, c) * (b, d)) = \boxtimes(a, c) * \boxtimes(b, d).$$

Unitality follows by applying π_1 to the composition $G \times \mathrm{pt} \xrightarrow{id \times e} G \times G \xrightarrow{m} G$, which is the identity. Thus, the premise of the Eckman-Hilton argument holds.

Another great proof I saw, which I didn't think of myself, was that a path $\alpha\colon I \to G$ defines a homotopy $h_t(g) = \alpha_t g$ from id_G to itself. By HW7, this implies that $[\alpha]$ is in the center of $\pi_1(G)$.

With all that said, it is also beneficial to see explicitly why the argument works. Given a path $\alpha\colon I \to G$ and $s \in I$, define α_s as the reparametrization of α to $[\frac{s}{2}, \frac{s+1}{2}]$ We then have $\alpha * \beta = \alpha_0 \beta_1$ and $\beta * \alpha = \alpha_1 \beta_0$. Commuting those two elements is the homotopy $\alpha_s \beta_{1-s}$ which slides the two paths part each other. Such sliding motion is at the heart of the Eckman-Hilton argument, and can also be seen in the commutativity of π_2.

2 Fiber bundles

Fiber bundles are a generalization of covering spaces, where the fibers are not necessarily discrete.

Definition 2.1. The *trivial bundle* over B with fiber F is the projection $B \times F \to B$. A map $p\colon E \to B$ is a *fiber bundle* with fiber F if it is locally trivial, meaning that for all $x \in B$ there is some neighborhood $x \in U$ such that $p^{-1}(U) \simeq U \times F$, with $p^{-1}(U) \to U$ corresponding to the projection $U \times F \to U$.

Example 2.2. $S^1 \times I \to S^1$ is a trivial bundle. On the other hand, consider the Möbius strip M, with the projection $M \to S^1$. Locally, M and $S^1 \times I$ look the same, so M is a fiber bundle. However, M is not globally trivial.

In the context of homotopy theory, the crucial property of fiber bundles is that they have a homotopy lifting property: For every homotopy $h\colon X \times I \to B$, and a lift of one of the edges $\tilde{h}_0\colon X \to E$ satisfying $p \circ \tilde{h}_0 = h_0$, there exists a (non-uniqe) lift $\tilde{h}\colon X \times I \to E$ such that $p \circ \tilde{h} = h$ and $\tilde{h}|_0 = \tilde{h}_0$. Any map $p\colon E \to B$ satisfying this homotopy lifting property is called a *fibration*. Such a fibration, with fiber F, is written akin to short exact sequences as

$$F \hookrightarrow E \to B.$$

Theorem 2.3. Given a fibration $F \hookrightarrow E \to B$, there is a long exact sequence in homotopy groups

$$\cdots \to \pi_n(F) \to \pi_n(E) \to \pi_n(B) \to \pi_{n-1}(F) \to \cdots$$

Let $U(n) \subseteq M_{n\times n}(\mathbb{C})$ be the group of unitary matrices, inheriting a topology from $\mathbb{C}^{n^2}$. Consider that map $p\colon U(n) \to S^{2n-1}$ which sends $A \in U(n)$, the last column $A_n \in S^{2n-1}$, which is a unit vector. Given $v \in S^{2n-1}$, let $v^\perp \simeq \mathbb{C}^{n-1}$ be its orthogonal complement. The fiber $p^{-1}(v)$ consists of unitary operators A that send $e_n \mapsto v$; in particular, they restrict to (and are determined by) unitary operators $A|_{\mathbb{C}^{n-1}}\colon \mathbb{C}^{n-1} \to v^\perp$. If we choose an orthonormal basis B for $v^\perp$ to represent $A|_{\mathbb{C}^{n-1}}$, and use the standard basis for $\mathbb{C}^{n-1}$, then we get that $[A|_{\mathbb{C}^{n-1}}]_B \in U(n-1)$. This gives us a homeomorphism $p^{-1}(v) \simeq U(n-1)$.

Proposition 2.4. *$p\colon U(n) \to S^{2n-1}$ is a fiber bundle, with fiber $U(n-1)$.*

Proof. Let $v \in S^{2n-1}$. Given a small enough neighborhood $v \in U$, we will choose for every $u \in U$ an orthonormal basis $B_u = (b_1(u), \dots, b_n(u))$ of $u^\perp$. This choice will be made continuously, meaning that $b_i\colon U \to S^{2n-1}$ are continuous. Using this basis, we will get a local trivialization $p^{-1}(U) \simeq S^{2n-1} \times U(n-1)$ by $A \mapsto (p(A), [A|_{\mathbb{C}^{n-1}}]_{B_{p(A)}})$.

Start with an orthonormal basis $B_v = (b_1, \dots, b_n)$, for any $u \in S^{2n-1}$ project B_v to $u^\perp$. The projected B_v is not always linearly independent – e.g. if $u = b_i$, then the projection of b_i is 0. However, it will be a basis for u close enough to v. Formally, consider the function $d\colon S^{2n-1} \to \mathbb{C}$ which sends $u \in S^{2n-1}$ to the determinant

$$d(u) = \det \begin{pmatrix} P_{u^\perp}(b_1) & \dots & P_{u^\perp}(b_{n-1}) & v \\ \downarrow & & \downarrow & \downarrow \end{pmatrix}$$

All the components of the above function are continuous, namely the determinant is a polynomial and the projection has a formula using the inner product. Moreover, $(b_1, \dots, b_{n-1}, v)$ forms an orthonormal basis, so $d(v) \neq 0$. Thus, we can choose the open neighborhood $v \in U = d^{-1}(\mathbb{C} \setminus \{0\})$, and for every $u \in U$ the projections $(P_{u^\perp}(b_1), \dots, P_{u^\perp}(b_{n-1}))$ will be linearly independent. Applying the Graham-Schmidt algorithm on the projections, which is also continuous, we get an orthonormal basis B_u of $u^\perp$. □

Corollary 2.5. *$\pi_k(U(n-1)) \to \pi_k(U_n)$ is an isomorphism for $k < 2n-1$, and a surjection for $k = 2n-1$.*

Proof. Follows from the fibration $U(n-1) \hookrightarrow U(n) \to S^{2n-1}$, and the fact that $\pi_k(S^{2n-1}) = 0$ for $k < 2n-1$. $\square$

Note that, for $v \in S^{2n-1}$, $v^{\perp}$ is the tangent space of S^{2n-1} at v. A global, continuous choice of a basis for the tangent spaces is called a *framing*. If we had a framing of S^{2n-1}, we would get a trivialization of the fiber bundle $U(n) \simeq S^{2n-1} \times U(n-1)$. This happens for $n = 1$, where indeed $U(1) \simeq S^1 \times U(0) = S^1$ and also less trivially for $n = 2$ where $U(2) \simeq SU(2) \times U(1) \simeq S^3 \times U(1)$. It is a very non-trivial fact, due to Hirzebuch-Kervaiere and Bott-Milnor, that such framings exist only for S^1, S^3 and S^7.

3 More Hopf maps

The Hopf map is a fiber-bundle $\eta \colon S^3 \to S^2$ with fiber S^1, corresponding to a non-trivial element in $\pi_3(S^2)$. To define the Hopf map, consider S^3, as the unit sphere in $\mathbb{C}^2$, and consider the quotient map $S^3 \to \mathbb{CP}^1 = S^3/x \sim \lambda x$. There is a homeomorphism $\mathbb{CP}^1 \simeq S^2$, and the fiber of the quotient at $[x] \in \mathbb{CP}^1$ is $\{\lambda x \mid \lambda \in S^1\} \simeq S^1$.

For this construction, we used only a small part of the structure of $\mathbb{C}$. In fact, we only used the fact that it is a normed division algebra.

Definition 3.1. A *normed division algebra* is a finite dimensional, normed real vector space V with a map $\cdot \colon V \times V \to V$ that is:

(1) Unital: there is an element $1 \in V$ such that $1 \cdot a = a \cdot 1 = a$.

(2) Normed: $||a \cdot b|| = ||a|| \cdot ||b||$

(3) Division: every $a \neq 0$ has a left inverse $a^{-1}a = 1$ (it follows that there is also a right inverse, but they need not coincide).

A theorem by Hurwitz (not to be confused with Hurewicz) states that there are only 4 normed division algebras, of dimensions 1,2,4,8:

(1) $\mathbb{R}$, which is an ordered field.

(2) $\mathbb{C}$, which is not ordered but still a field.

(3) $\mathbb{H}$, the quaternions, which are not commutative but still associative.

(4) $\mathbb{O}$, the octonions, which are not even associative.

Let V be a normed division algebra of dimension n. Consider $S(V)$ the unit sphere in V. As a normed vector space, V is equivalent to $\mathbb{R}^n$ with the Euclidean norm, and in particular $S(V) \simeq S^{n-1}$. Similarly, V^2 has a norm given by $||(x, y)|| = \sqrt{||x||^2 + ||y||^2}$, and the unit sphere there is $S(V^2) \simeq S^{2n-1}$. Define $P(V^2)$ as the space of lines in V^2, or formally as one of the following quotients:

$$P(V^2) = V^2 - 0/(x, y) \sim (\lambda x, \lambda y) \; \forall \lambda \in V - 0.$$

$$P(V^2) = S(V^2)/(x, y) \sim (\lambda x, \lambda y) \; \forall \lambda \in S(V).$$

The fact that the two definitions are well-defined, and are equivalent, follows from the multiplication of V being normed.

Lemma 3.2. *There is a homeomorphism $P(V^2) \simeq S^n$.*

Proof. Let $V_+ = V \cup \{\infty\} \simeq S^n$ be the one point compactification of V. Consider the map $P(V^2) \to V_+$ which sends

$$[x, y] \mapsto \begin{cases} x^{-1}y & x \neq 0 \\ \infty & x = 0 \end{cases}$$

This map is well-defined, because given $[x, y] = [\lambda x, \lambda y]$ we have $(\lambda x)^{-1}\lambda y = x^{-1}\lambda^{-1}\lambda y = x^{-1}y$. In the other direction, consider the map $V_+ \to P(V^2)$ which sends

$$z \mapsto \begin{cases} [1, z] & z \neq \infty \\ [0, 1] & z = \infty \end{cases}$$

Those maps are inverses, as $[x, y] = [1, x^{-1}y]$ for $x \neq 0$ and $[0, y] = [0, 1]$. The proof that those maps are continuous is exactly as in the cases $\mathbb{R}$ or $\mathbb{C}$. □

Definition 3.3. Given a normed division algebra V of dimension n, there is a corresponding generalized Hopf map $\eta_V : S^{2n-1} \to S^n$, which is the quotient map $S(V^2) \to P(V^2)$.

η_V is a fiber bundle, with the fiber at $[x, y]$ given by $\{(\lambda x, \lambda y) | \lambda \in S(V)\} \simeq S^{n-1}$. The proof that $\eta = \eta_{\mathbb{C}}$ defined a non-trivial element in $\pi_3(S^2)$ used the fibration $S^1 \hookrightarrow S^3 \to S^2$, and relied on the vanishing of $\pi_k(S^1) = 0$ for $k > 1$. While this proof does not work for $n > 2$, it is still true that $[\eta_V] \in \pi_{2n-1}(S^{n-1})$ is non-trivial. Given that there are only four options for V, we get 4 non-trivial elements in the homotopy groups of the spheres, and also 4 fibrations of sphere:

(1) $S^0 \hookrightarrow S^1 \xrightarrow{\eta_{\mathbb{R}}} S^1$

(2) $S^1 \hookrightarrow S^3 \xrightarrow{\eta_{\mathbb{C}}} S^2$

(3) $S^3 \hookrightarrow S^7 \xrightarrow{\eta_{\mathbb{H}}} S^4$

(4) $S^7 \hookrightarrow S^{15} \xrightarrow{\eta_{\mathbb{O}}} S^8$

A theorem by Adams says that those are the only fibrations where both the base, the total space and the fiber are spheres.