Algebraic topology - Recitation 12

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1 Exact functors

Let X be a space and A an Abelian group. In class, you defined homology with coefficients in A. To do so, you constructed a chain complex $\mathrm{C}_\bullet(X;A)$ by tensoring $\mathrm{C}_\bullet(X)$ with A:

$$\cdots \mathrm{C}_2(X)\otimes A \xrightarrow{\partial_2\otimes A} \mathrm{C}_1(X)\otimes A \xrightarrow{\partial_1\otimes A} \mathrm{C}_0(X)\otimes A\cdots$$

and defined $\mathrm{H}_n(X;A)=\mathrm{H}_n(C_\bullet(X;A))$. It would have been very simple if we had an isomorphism $\mathrm{H}_n(X;A)\simeq \mathrm{H}_n(X)\otimes A$. To see why this is not the case, let us try to prove that it is until we get stuck. By definition,

$$\mathrm{H}_n(X;A)=\ker(\partial_n\otimes A)/\operatorname{Im}(\partial_{n+1}\otimes A)=\operatorname{coker}(\mathrm{C}_{n+1}(X)\otimes A\xrightarrow{\partial_{n+1}\otimes A}\ker(\partial_n\otimes A))$$

$$\mathrm{H}_n(X)\otimes A=\operatorname{coker}(\mathrm{C}_{n+1}(X)\xrightarrow{\partial_{n+1}}\ker(\partial_n))\otimes A$$

Thus, if we had $\ker(d\otimes A)\simeq\ker(d)\otimes A$ and $\operatorname{coker}(d\otimes A)\simeq\operatorname{coker}(d)\otimes A$, then we would deduce $\mathrm{H}_n(X;A)\simeq\mathrm{H}_n(C_\bullet(X;A))$.

Definition 1.1. A functor $F\colon \mathrm{Ab}\to\mathrm{Ab}$ is called:

- *left exact* if F preserves ker, or equivalently every left short exact sequence $0\to A\to B\to C$ is sent to a left short exact sequence $0\to F(A)\to F(B)\to F(C)$ (a left short exact sequence is of the form $0\to\ker(f)\to B\xrightarrow{f}C$ for arbitrary f),
- *right exact* if F preserves coker, or equivalently every right short exact sequence $A\to B\to C\to 0$ is sent to a right short exact sequence $F(A)\to F(B)\to F(C)\to 0$,
- *exact* if F is both left and right exact, or equivalently every short exact sequence $0\to A\to B\to C\to 0$ is sent to a short exact sequence $0\to F(A)\to F(B)\to F(C)\to 0$.

Whether a functor F commutes with homology is related to the exactness of F.

Lemma 1.2. *Let $F\colon\mathrm{Ab}\to\mathrm{Ab}$ be a functor and C a chain complex.*

(1) *If f is left exact, then there is a map $\mathrm{H}_n(F(C))\to F(\mathrm{H}_n(C))$.*

(2) *If f is right exact, then there is a map $F(\mathrm{H}_n(C))\to\mathrm{H}_n(F(C))$.*

(3) *If f is exact, then there is an isomorphism $F(\mathrm{H}_n(C)) \simeq \mathrm{H}_n(F(C))$.*

Proof. (1) Suppose F is left exact, so it preserves kernels. We have

$$\mathrm{H}_n(F(C)) = \operatorname{coker}(F(\mathrm{C}_{n+1}(X)) \xrightarrow{F(\partial_{n+1})} \ker(F(\partial_n))) \simeq \operatorname{coker}(F(\mathrm{C}_{n+1}(X)) \xrightarrow{F(\partial_{n+1})} F(\ker(\partial_n))).$$

Cokernels are colimits, so even if F does not preserve them there is always an assembly map

$$\operatorname{coker}(F(\mathrm{C}_{n+1}(X)) \xrightarrow{F(\partial_{n+1})} F(\ker(\partial_n))) \to F(\operatorname{coker}(\mathrm{C}_{n+1}(X) \xrightarrow{\partial_{n+1}} \ker(\partial_n))).$$

Explicitly, this is a map

$$F(\ker(\partial_n))/\operatorname{Im}(F(\partial_{n+1})) \to F(\ker(\partial_n)/\operatorname{Im}(\partial_{n+1}))$$

which comes from $F(\ker(\partial_n)) \to F(\ker(\partial_n)/\operatorname{Im}(\partial_{n+1}))$, where we notice that the composition

$$F(\operatorname{Im}(\partial_{n+1})) \to F(\ker(\partial_n)) \to F(\ker(\partial_n)/\operatorname{Im}(\partial_{n+1}))$$

is 0. This is a map $\mathrm{H}_n(F(C)) \to F(\mathrm{H}_n(C))$.

(2) Suppose F is right exact, so it preserves cokernels. The kernel is a limit, so even if F does not preserve it there is always an assembly map (in the other direction) $F(\ker(\partial_n)) \to \ker(F(\partial_n)$. Explicitly, this comes from the map $F(\ker(\partial_n)) \to F(C_n)$, noticing that the composition

$$F(\ker(\partial_n)) \to F(C_n) \to F(C_{n-1})$$

is 0. Applying cokernel to this map, we get by functoriality

$$\begin{aligned} F(\mathrm{H}_n(C)) = F(\operatorname{coker}(\mathrm{C}_{n+1}(X) \xrightarrow{\partial_{n+1}} \ker(\partial_n))) &\simeq \operatorname{coker}(F(\mathrm{C}_{n+1}(X)) \xrightarrow{F(\partial_{n+1})} F(\ker(\partial_n))) \\ &\to \operatorname{coker}(F(\mathrm{C}_{n+1}(X)) \xrightarrow{F(\partial_{n+1})} \ker(F(\partial_n)))) = \mathrm{H}_n(F(C)) \end{aligned}$$

(3) If F is exact, then it preserves kernels and cokernels, so it preserves homology.

□

So, is the functor $-\otimes A\colon \mathrm{Ab} \to \mathrm{Ab}$ exact?

Proposition 1.3. *$-\otimes A$ is always right exact, but not always left exact.*

Proof. To see that $-\otimes A$ is right exact, the idea is that it is a left adjoint $-\otimes A \dashv \hom(A,-)$ so it preserves all colimits, in particular coker. Explicitly, consider $f\colon B \to C$. The cokernel $\operatorname{coker}(f) = C/\operatorname{Im}(f)$ has the universal property that homomorphisms $\operatorname{coker}(f) \to D$ correspond to homomorphisms $g\colon C \to D$ such that $g \circ f = 0$. Similarly, homomorphisms $\operatorname{coker}(f \otimes A) \to D$ correspond to homomorphisms $g\colon C \otimes A \to D$ such that $g \circ (f \otimes A) = 0$. However, a homomorphism $g\colon C \otimes A \to D$ corresponds to a homomorphism into the internal hom $\bar{g}\colon C \to \hom(A, D)$. Under this identification, the fact that the composition

$$B \otimes A \xrightarrow{f\otimes A} C \otimes A \xrightarrow{g} D$$

is 0 corresponds to the claim that the composition

$$B \xrightarrow{f} C \xrightarrow{\bar{g}} \hom(A, D)$$

is 0. However, this corresponds to a homomorphism $\operatorname{coker}(f) \to \hom(A, D)$, which corresponds to a homomorphism $\operatorname{coker}(f) \otimes A \to D$. Thus, $\operatorname{coker}(f \otimes A)$ and $\operatorname{coker}(f) \otimes A$ have the same universal property, so they are isomorphic.

For an example that is not left exact, consider $A = \mathbb{Z}/2$ and the left short exact sequence $0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}/2$. Tensoring with $\mathbb{Z}/2$ amounts to dividing by 2, so we get the sequence $0 \to \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \xrightarrow{\mathrm{id}} \mathbb{Z}/2$ which is not exact. □

Remark 1.4. If F is right exact and C is non-negatively-graded, i.e. $C_i = 0$ for $i < 0$, then $F(\mathrm{H}_0(C)) \to \mathrm{H}_0(F(C))$ is an isomorphism. This is because $\mathrm{H}_0(C) = C_0 / \operatorname{Im}(\partial_1) = \operatorname{coker}(\partial_1)$, and no kernels are involved. In particular, for $F = - \otimes A$ and $C = \mathrm{C}_\bullet(X)$, we get $\mathrm{H}_0(X) \otimes A \simeq \mathrm{H}_0(X; A)$.

While $- \otimes A$ is generally not left exact, it is for some special A.

Lemma 1.5. *If A is a free Abelian group, then $- \otimes A$ is exact.*

Proof. Write $A = \mathbb{Z}\langle S \rangle$, it follows that $B \otimes A = B\langle S \rangle$. Consider $f\colon B \to C$, the induces $f \otimes A\colon B\langle S \rangle \to C\langle S \rangle$ maps $\sum_i b_i s_i = \sum_i f(b_i) s_i$. The kernel $\ker(f \otimes A)$ consists of $\sum_i b_i s_i$ such that $\sum_i f(b_i) s_i = 0$, but because it is a formal sum this is equivalent to $f(b_i) = 0$, i.e. $b_i \in \ker(f)$. This gives as an equivalence $\ker(f \otimes A) \simeq \ker(f)\langle S \rangle = \ker(f) \otimes A$. □

We conclude by saying that, while there is a map $\mathrm{H}_n(X) \otimes A \to \mathrm{H}_n(X; A)$, it is not necessarily an isomorphism. To quantify how far away it is from an isomorphism, we need to a way to quantify how far away $- \otimes A$ is from being exact. This is the goal for the remainder today.

2 Tensor product of chain complexes

Let C, D be chain complexes. We can define a bicomplex $(C \boxtimes D)_{i,j} = C_i \otimes D_j$, with horizontal boundaries coming from C and vertical boundaries coming from D (this is the commuting version of bicomplexes, we can add minus signs when we want to take spectral sequences).

Definition 2.1. The tensor product of chain complexes is defined as the total complex $C \otimes D := \operatorname{Tot}(C \boxtimes D)$. That is, $(C \otimes D)_n = \bigoplus_{i+j=n} C_i \otimes D_j$.

Example 2.2. For $A \in \mathrm{Ab}$, let $\underline{A} \in \mathrm{Ch}$ be the chain supported on degree 0:

$$\cdots \to 0 \to A \to 0 \to \cdots .$$

For any other chain C, we get that $(C \otimes \underline{A})_n = C_n \otimes A$.

The operation $C \otimes - \colon \mathrm{Ch} \to \mathrm{Ch}$ is a functor, and in particular sends isomorphisms to isomorphisms. However, the main thing we care about in a chain complex is its homology, so we care about a weaker notion of isomorphism.

Definition 2.3. A chain map $C \to D$ is called a *quasi-isomorphism* is the induced map on homology $\mathrm{H}_n(C) \to \mathrm{H}_n(D)$ is an isomorphism. If $X \to Y$ is a homotopy equivalence of spaces, then $\mathrm{C}_\bullet(X) \to \mathrm{C}_\bullet(Y)$ is a quasi-isomorphism

The functor $C \otimes -$ does not preserve quasi-isomorphisms in general, for essentialy the same reason that tensoring is not always exact.

Example 2.4. Consider the quasi-isomorphism

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r, "\times 2"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}/2 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

with $\mathrm{H}_0 = \mathbb{Z}/2$ and $\mathrm{H}_i = 0$ for $i \neq 0$. Tensoring with $\underline{\mathbb{Z}/2}$, we get

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{Z}/2 \arrow[r, "0"] \arrow[d] & \mathbb{Z}/2 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}/2 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

which is no longer a quasi-isomorphism, as the top row has non-trivial $\mathrm{H}_1 = \mathbb{Z}/2$.

In some cases $C \otimes -$ does preserve quasi-isomorphisms. To detect that, we will use spectral sequences.

2.1 Interlude on spectral sequences

Let C be a chain complex with an infinite increasing filtration $0 = C^{(0)} \subseteq C^{(1)} \subseteq \cdots \subseteq C$. We say that the associated spectral sequence *converges* if $C = \bigcup C^{(s)}$ and $E^r_{n,s}$ stabilizes locally. In HW6 you proved that in this case, $E^\infty_{n,s} = \mathrm{H}^{(s)}_n(C)/\mathrm{H}^{(s-1)}_n(C)$.

Lemma 2.5. *Let C, D be chain complexes with converging filtrations $C^{(s)}, D^{(s)}$, and suppose $f\colon C \to D$ is a chain map such that $f(C^{(s)}) \subseteq D^{(s)}$. In particular f induces a map on spectral sequences $E^r_{n,s}(C) \to E^r_{n,s}(D)$. If for some r this map is an isomorphism, then f is a quasi-isomorphism.*

Proof. If f is an isomorphism for some r, then by turning the pages it is an isomorphism for every $r' > r$, and in particular for ∞. From convergence, we get that f induces isomorphisms $\mathrm{H}^{(s)}_n(C)/\mathrm{H}^{(s-1)}_n(C) \xrightarrow{\sim} \mathrm{H}^{(s)}_n(D)/\mathrm{H}^{(s-1)}_n(D)$. For $s = 0$ we get that $\mathrm{H}^{(0)}_n(C) \xrightarrow{\sim} \mathrm{H}^{(0)}_n(D)$, and by induction we get $\mathrm{H}^{(s)}_n(C) \xrightarrow{\sim} \mathrm{H}^{(s)}_n(D)$. This can be seen from the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{H}^{(s-1)}_n(C) \arrow[r] \arrow[d, "\sim"] & \mathrm{H}^{(s)}_n(C) \arrow[r] \arrow[d] & \mathrm{H}^{(s)}_n(C)/\mathrm{H}^{(s-1)}_n(C) \arrow[r] \arrow[d, "\sim"] & 0 \\
0 \arrow[r] & \mathrm{H}^{(s-1)}_n(D) \arrow[r] & \mathrm{H}^{(s)}_n(D) \arrow[r] & \mathrm{H}^{(s)}_n(D)/\mathrm{H}^{(s-1)}_n(D) \arrow[r] & 0
\end{tikzcd}$$

And we have $\mathrm{H}_n(C) = \bigcup_s \mathrm{H}_n^{(s)}(C)$ and $\mathrm{H}_n(D) = \bigcup_s \mathrm{H}_n^{(s)}(D)$, so we get an isomorphism $\mathrm{H}_n(C) \to \mathrm{H}_n(D)$. ☐

The example we will apply this to is the horizontal filtration of a bicomplex,

Example 2.6. Suppose B is a non-negatively-graded bicomplex, meaning $B_{i,j} = 0$ if $i < 0$ or $j < 0$. The horizontal filtration was defined as

$$\mathrm{Tot}(B)_n^{<(s)} = \bigoplus_{0 \leq i \leq s} B_{n-i,i}$$

The E^0 page is $B_{n-s,s}$

$$\begin{array}{ccccccc} & & B_{0,1} & \longleftarrow & B_{1,1} & \longleftarrow & B_{2,1} \\ & & & & & & \\ B_{0,0} & \longleftarrow & B_{1,0} & \longleftarrow & B_{2,0} & & \end{array}$$

and the E^1 page is $\mathrm{H}_n(B_{\bullet-s,s})$, with the differentials going down. Because we have the trivial group below the zero horizontal line and above the diagonal line, every differential will eventually hit 0, so this spectral sequence converges.

2.2 Preserving quasi-isomorphisms

Recall that we were asking when tensoring with a chain complex preserves quasi-isomorphism.

Proposition 2.7. *Suppose C, D, D' are non-negatively-graded chain complexes, such that $f\colon D \to D'$ is a quasi-isomorphism and C_i are free Abelian groups. Then $C \otimes f\colon C \otimes D \to C \otimes D'$ is a quasi-isomorphism.*

Proof. Consider the vertical filtrations on $C \otimes D = \mathrm{Tot}(C \boxtimes D)$ and $C' \otimes D = \mathrm{Tot}(C' \boxtimes D)$. f induces a map on the E^1-pages

$$E^1_{n,s}(C \otimes D) = \mathrm{H}_n(C_{\bullet-s} \otimes D_s) \to \mathrm{H}_n(C_{\bullet-s} \otimes D'_s) = E^1_{n,s}(C \otimes D')$$

This map is an isomorphism, because tensoring with a free Abelian group is exact. It follows that $\mathrm{H}_n(C \otimes D) \to \mathrm{H}_n(C \otimes D')$ is an isomorphism. ☐

3 Free resolution and Tor

We saw that tensoring with a chain complex of free Abelian groups is well-behaved. The process of replacing a chain complex with a quasi-isomorphic chain complex of free Abelian groups is called *free resolution*. We will only consider free resolutions of $\underline{A}$.

Definition 3.1. Let $A \in \mathrm{Ab}$, a free resolution of A is a non-negatively graded chain complex P of free Abelian groups and a quasi-isomorphism $P \to \underline{A}$:

$$\begin{tikzcd} \cdots \arrow[r] & P_2 \arrow[r] \arrow[d] & P_1 \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d, "f"] & 0 \arrow[r] & \cdots \\ \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & A \arrow[r] & 0 \arrow[r] & \cdots \end{tikzcd}$$

A free resolution always exists. In fact, we can always choose it to be of length 2. Let A be an Abelian group, and suppose $S \subseteq A$ is a set of generators (e.g. $S = A$), then we have a surjective map $f\colon \mathbb{Z}\langle S\rangle \to A$. Every subgroup of a free Abelian group is free, so $\ker(f) \subseteq \mathbb{Z}\langle S\rangle$ is also free.

Lemma 3.2.

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & \ker(f) \arrow[r] \arrow[d] & \mathbb{Z}\langle S\rangle \arrow[r] \arrow[d, "f"] & 0 \arrow[r] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & A \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

is a free resolution of A.

Proof. The 0-homology of the top row is $\mathbb{Z}\langle S\rangle / \ker(f) \simeq A$, and the other homologies are 0. □

Remark 3.3. If we worked with modules over a ring instead of Abelian groups, we will still have free resolutions, but they might be of infinite length, e.g. for modules over $\mathbb{Z}[x]$. For modules over a field, i.e. vector spaces, every vector space is free, so there are free resolutions of length 1.

Definition 3.4. Let A be an Abelian group with a free resolution $P \to \underline{A}$. For any other Abelian group B, define $\mathrm{Tor}_n(A, B) = \mathrm{H}_n(P \otimes B)$.

Proposition 3.5. $\mathrm{Tor}_n(B, A)$ *is well-defined, i.e. does not depend on the free resolution P.*

Proof. Suppose P, P' are free resolutions of A, and let Q be a free resolution of B. Because non-negatively-graded free chain complexes preserve quasi-isomorphisms, we get

$$\mathrm{H}_n(P \otimes B) \simeq \mathrm{H}_n(P \otimes Q) \simeq \mathrm{H}_n(A \otimes Q) \simeq \mathrm{H}_n(P' \otimes Q) \simeq \mathrm{H}_n(P' \otimes B)$$

□

If we take P to be a free resolution of length 2, we see that $\mathrm{Tor}_n(A, B) = 0$ for $n \geq 2$. Also, we know that tensoring with B commutes with H_0, so $\mathrm{Tor}_0(A, B) = \mathrm{H}_0(P \otimes B) \simeq \mathrm{H}_0(P) \otimes B \simeq A \otimes B$. Thus, only $\mathrm{Tor}_1(A, B)$ is non-trivial. The reason we care about Tor is that it measures how far $A \otimes -$ is from being left exact.

Proposition 3.6. *Suppose* $0 \to B \to C \to D \to 0$ *is a short exact sequence of Abelian groups, then*

$$0 \to \mathrm{Tor}_1(A, B) \to \mathrm{Tor}_1(A, C) \to \mathrm{Tor}_1(A, D) \to A \otimes B \to A \otimes C \to A \otimes D \to 0$$

is exact.

Proof. In homework. □

The construction of Tor and the proof that it is well-defined required some sophisticated machinery, but calculating Tor is pretty straightforward. Let us see this in a couple of example.

Example 3.7. Let A be a free Abelian group, then $\mathrm{Tor}_1(A, B) = 0$. This is because $\underline{A}$ is a free resolution of itself, so $\mathrm{Tor}_1(A, B) = \mathrm{H}_1(\underline{A \otimes B}) = 0$.

Example 3.8. Let $A = \mathbb{Z}/n$, then $\mathrm{Tor}_1(\mathbb{Z}/n, B) = \{b \in B | nb = 0\}$ is the subgroup of n-torsion elements (this is the origin of the name). Consider the free resolution of $\mathbb{Z}/2$ given by $P = \cdots \to 0 \to \mathbb{Z} \xrightarrow{\times n} \mathbb{Z}$, we get that $P \otimes B = \cdots \to 0 \to B \xrightarrow{\times n} B$, so $\mathrm{H}_1(P \otimes B) = \ker(B \xrightarrow{\times n} B)$.

Example 3.9. Let $A = A_1 \oplus A_2$, then $\mathrm{Tor}_1(A_1 \oplus A_2, B) \simeq \mathrm{Tor}_1(A_1, B) \oplus \mathrm{Tor}_1(A_2 \oplus B)$. Suppose P_1, P_2 are free resolutions of A_1, A_2, then $P_1 \oplus P_2$ is a free resolution of $A_1 \oplus A_2$. Thus

$$\mathrm{H}_1((P_1 \oplus P_2) \otimes B) \simeq \mathrm{H}_1(P_1 \otimes B \oplus P_2 \otimes B) \simeq \mathrm{H}_1(P_1 \otimes B) \oplus \mathrm{H}_1(P_2 \otimes B).$$

4 Ext

Up to now, we considered the operation $A \otimes -$. An equally important operation is its right adjoint, $\hom(A, -)$.

Lemma 4.1. $\hom(A, -)$ *is left exact, but not necessarily right exact.*

Proof. It is a right adjoint, so it preserves all limits, and in particular ker. To see an example where it is not right exact, take the same old example $\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$, which is sent by $\hom(\mathbb{Z}/2, -)$ to $0 \to 0 \to \mathbb{Z}/2 \to 0$, which is not exact. $\square$

To quantify how far $\hom(A, -)$ is from being right exact, we define the Ext groups:

Definition 4.2. Let A be an Abelian group with free resolution $P_\bullet$, and let B be another Abelian group. Then we get a cochain complex

$$0 \to \hom(P_0, B) \to \hom(P_1, B) \to \dots,$$

and we define Ext as it's cohomology $\mathrm{Ext}^n(A, B) = \mathrm{H}^n(\hom(P_\bullet, B))$.

As before, we get that $\mathrm{Ext}^0(A, B) = \hom(A, B)$ and that $\mathrm{Ext}^n(A, B) = 0$ for $n \geq 2$, so we only care about Ext^1.

Proposition 4.3. *Suppose* $0 \to B \to C \to D \to 0$ *is a short exact sequence of Abelian groups, then*

$$0 \to \hom(A, B) \to \hom(A, C) \to \hom(A, D) \to \mathrm{Ext}^1(A, B) \to \mathrm{Ext}^1(A, C) \to \mathrm{Ext}^1(A, D) \to 0$$

is exact.