Algebraic topology - Recitation 13

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1 Cohomology

We already saw two ways of recognizing holes in a space X:

(1) Homotopy: a hole is a m ap $S^n \to X$ which is not null-homotopic.

(2) Homology: a hole is a cycle that is not a boundary.

We now have a third invariant, cohomology, which can also claim to recognize holes. To see how, we will consider the simple example of an oriented graph.

1.1 Cohomology of a graph

Let A be some Abelian group, and let $X = (V, E)$ be an oriented graph, modeled either as a 1-dimensional semi-simplicial set or a 1-dimensional CW-complex. The cochain complex $\mathrm{C}^\bullet(X; A)$ is described as follows:

- The 0-cochains $\mathrm{C}^0(X; A) = \hom(\mathrm{C}_0(X), A) = \hom(\mathbb{Z}\langle V\rangle, A)$ are maps $V \to A$.
- The 1-cochains $\mathrm{C}^1(X; A) = \hom(\mathrm{C}_1(X), A) = \hom(\mathbb{Z}\langle E\rangle, A)$ are maps $E \to A$.
- The differential $d\colon \mathrm{C}^0(X; A) \to \mathrm{C}^1(X; A)$ takes a 0-cochain ϕ to the 1-cochain $d\phi = \phi \circ \partial$. Explicitly, if e is an edge between vertices e_0 and e_1, then $d\phi(e) = \phi(\partial(e)) = \phi(e_1 - e_0) = \phi(e_1) - \phi(e_0)$.

Think of X as the graph of trails on a hill, where the edges are paths and the vertices are intersection. Alternatively, think of X as an electrical circuit with wires connecting nodes. Taking $A = \mathbb{R}$, we can think of a 0-cochain ϕ as measuring the elevation at each intersection or the voltage at each node, and $d\phi$ tells us the elevation gain of a path or the voltage difference on a wire.

A 0-cocycle is then a 0-cochain ϕ such that $d\phi = 0$. It is an elevation map where all paths conserve the elevation, or all wires conserve the voltage. If ϕ is a 0-cocycle and a and b are connected, then in particular $\phi(a) = \phi(b)$. It follows that a 0-cocylce is determined by the (constant) values that it has on each connected component. In degree 0 we have no coboundaries, so $\mathrm{H}^0(X; A) = \ker(d) \simeq A^{\pi_0(X)}$, a product of A's with a copy for each connected component.

A 1-coboundary is a 1-cochain ψ which is of the form $\psi = d\phi$. In a graph every 1-cochain will be a 1-cocycle, so $\mathrm{H}^1(X;A) = \mathrm{C}^1(X;A)/\operatorname{Im}(d)$. Saying that $\mathrm{H}^1(X;A)$ is trivial amount to saying that for every 1-cochain ψ the equation $\psi = d\phi$ has a solution. This is analogous to finding an indefinite integral of a function; as with integrals, if a solution exists then it is unique up to a constant term in each connected component, namely up to a 0-cocyle.

If X is a tree, then the equation $\psi = d\phi$ can always be solved: start by choosing a root vetrex r and setting $\phi(r) = 0$, and for any other vertex a with a (unique) path $(e_1, \dots e_n)$ from r to a define $\phi(a) = \psi(e_1) + \cdots + \psi(e_n)$ (if X is not connected, choose a root for every connected component).

If X is not a tree, then we can choose a spanning tree T, and use T to define ϕ as above. In order for $\psi = d\phi$ to hold, we need $\psi(e) = \phi(e_1) - \phi(e_0)$ for every edge $e \in X \setminus T$. Since ψ can have an arbitrary value on each such edge, we see that $\mathrm{H}^1(X;A)$ is the product of A's with a copy for every edge not in T. Every such edge creates a hole. Note that the relationship between $\mathrm{H}^1(X;A)$ and $\mathrm{H}_1(X;A)$ is similar to the relationship between $\mathrm{H}^0(X;A)$ and $\mathrm{H}_0(X;A)$; in both cases we replaced the direct sum (coproduct) of copies of A with a product of copies of A.

1.2 A taste of de Rham

The analogy between cohomology and concepts like derivatives and integrals can be made precise; this is done with de Rahm cohomology, which you will learn about if you take differential geometry next semester.

To give you a taste of those ideas, consider $X = \mathbb{C} - \{0\}$. We will prove directly, using singular cohomology, that $\mathrm{H}^1(X;\mathbb{C}) \neq 0$, namely that there exists a 1-cocylce which is not a 1-coboundary. Define a 1-cochain ψ as follows: for every 1-simplex, which is a continuous path $\gamma\colon \Delta^1 \to X$, define $\psi(\gamma) = \int_\gamma \frac{dz}{z}$. This ψ is a 1-cocylce, because for every $\sigma\colon \Delta^2 \to X$ we have that $\partial\sigma$ is a closed path that doesn't have 0 on its interior, so

$$d\psi(\sigma) = \oint_{\partial\sigma} \frac{dz}{z} = 0.$$

However, if ψ was a 1-coboundary, we would have a 0-cochain ϕ such that $\psi = d\phi$. A 0-cochain is a function on the 0-simplices, which are points $\Delta^0 \to X$. Thus, we can think of ϕ as a map $X \to \mathbb{C}$, and the fact that $\psi = d\phi$ translates to ϕ being a primitive function of $\frac{1}{z}$, which does not exist. Concretely, consider $\gamma\colon \Delta^1 \to X$ given by $\gamma(t) = e^{2\pi i t}$, then

$$\psi(\gamma) = \oint_\gamma \frac{dz}{z} = 2\pi i,$$

while the existence of $\psi = d\phi$ would imply $\psi(\gamma) = \phi(\gamma(1)) - \phi(\gamma(0)) = 0$.

2 Cup product

The main advantage of cohomology over homology is that it has an extra multiplicative structure.

Definition 2.1. A graded ring E^* is a sequnce of abelian groups E^i for $i \in \mathbb{Z}$ with a ring structure on $\bigoplus_i E^i$, such that if $a \in E^i$ and $b \in E^j$, then $ab \in E^{i+j}$. Given graded rings E^*, F^*, a graded ring map $f\colon E^* \to F^*$ is a collection of homomorphisms $E^i \to F^i$ that assemble to a ring homomorphisms $\oplus_i E^i \to \oplus_i F^i$. This defines a category grRing.

Example 2.2. The polynomial ring $\mathbb{Z}[x]$, where x is a generator in degree 1, is a graded ring. The degree n elements are the degree n homogenous polynomials $(\mathbb{Z}[x])^n = \mathbb{Z}\langle x^n \rangle$.

Let R be a ring, usually $\mathbb{Z}$ or a field, and let X be a space. The cohomology groups $\mathrm{H}^*(X;R)$ have a graded ring structure with multiplication given by the cup product: for $\phi \in \mathrm{H}^n(X;R)$ and $\psi \in \mathrm{H}^k(X;R)$, we have $\phi \smile \psi \in \mathrm{H}^{n+k}(X;R)$. This graded ring structure is functorial, meaning that homology is a functor $\mathrm{H}^*(-;R)\colon \mathrm{hTop} \to \mathrm{grRing}$

To describe the cup product on the level of cocycles, start with a semi-simplicial set X. There are boundary maps $(-)|_{[0,\dots,n]}\colon X_{n+k} \to X_n$ induced from the map $[n] \to [n+k]$ sending $i \mapsto i$, and $(-)|_{[n,\dots,n+k]}\colon X_{n+k} \to X_k$ induced from the map $[k] \to [n+k]$ sending $i \mapsto i+n$. Given cocycles $\phi \in \mathrm{C}^n(X;R)$ and $\psi \in \mathrm{C}^k(X;R)$, define $\phi \smile \psi \in \mathrm{C}^{n+k}(X;R)$ by mapping an $n+k$-simplex $\sigma \in X_{n+k}$ to

$$(\phi \smile \psi)(\sigma) = \phi(\sigma|_{[0,\dots,n]}) \cdot \psi(\sigma|_{[n,\dots,n+k]}) \in R$$

where the multiplication uses the ring structure of R.

2.1 The torus

To calculate the cup product on the torus, we first need to find the cohomology groups $\mathrm{H}^*(\mathbb{T},\mathbb{Z})$. For that we will use the CW-structure, with a single 0-cell, two 1-cells and a single 2-cell.

$$\begin{array}{ccc} v & \xrightarrow{\;a\;} & v \\ {\scriptstyle b}\downarrow & C & \downarrow{\scriptstyle b} \\ v & \xrightarrow{\;a\;} & v \end{array}$$

Remark 2.3. Given a finitely generated free abelian group $\mathbb{Z}\langle a_1,\dots,a_r\rangle$, then its dual group $\hom(\mathbb{Z}\langle a_1,\dots,a_r\rangle,\mathbb{Z})$ is also free of rank r, and is generated by $a_1^*,\dots,a_r^* \in \hom(\mathbb{Z}\langle a_1,\dots,a_r\rangle,\mathbb{Z})$ given by $a_i^*(a_j) = \delta_{ij}$.

The cellular chain complex is

$$\mathbb{Z}\langle v\rangle \xleftarrow{\partial_1} \mathbb{Z}\langle a,b\rangle \xleftarrow{\partial_2} \mathbb{Z}\langle C\rangle$$

where $\partial_2(C) = 0$ and $\partial_1(a) = \partial_1(b) = 0$. The cochain complex is

$$\mathbb{Z}\langle v^*\rangle \xrightarrow{d_1} \mathbb{Z}\langle a^*,b^*\rangle \xrightarrow{d_2} \mathbb{Z}\langle C^*\rangle$$

and $d_1 = d_2 = 0$. It follows that the cohomology is the same as the homology:

$$\mathrm{H}^*(\mathbb{T};\mathbb{Z}) \simeq \begin{cases} \mathbb{Z} & * = 0,2 \\ \mathbb{Z}^2 & * = 1 \\ 0 & \text{else} \end{cases}$$

To find the cup product, it is enough to consider how it acts on generators. Notice that $\mathrm{H}^n(\mathbb{T},\mathbb{Z}) = 0$ for $n > 2$, so we only need to find $x \smile y$ where the degrees of x and y add up to at most 2.

Proposition 2.4. *We have the following cup products:*

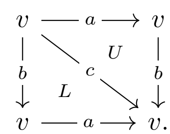
(1) $v^* \smile x = x \smile v^* = x$, *namely* v^* *is the unit of the ring.*

(2) $a^* \smile a^* = b^* \smile b^* = 0$

(3) $a^* \smile b^* = C^*$

(4) $b^* \smile a^* = -C^*$

Proof. To calculate the cup product, we will use the semi-simplicial set X which realizes to $\mathbb{T}$:



(1) Suppose $x \in \mathrm{H}^n(X;\mathbb{T})$ and let $\sigma \in \mathrm{C}^n(X;\mathbb{T})$.

$$v^* \smile x(\sigma) = v^*(\sigma|_{[0]})x(\sigma|_{[0,\dots n]}) = v^*(v)x(\sigma) = x(\sigma),$$

where we used the fact that v is the only 0-simplex so $\sigma|_{[0]} = v$. The calculation for $x \smile v^* = x$ is similar.

(2) It is enough to check $a^* \smile a^*$ on U and L.

$$(a^* \smile a^*)(U) = a^*(U|_{[0,1]})a^*(U|_{[1,2]}) = a^*(a)a^*(b) = 0$$

$$(a^* \smile a^*)(L) = a^*(L|_{[0,1]})a^*(L|_{[1,2]}) = a^*(b)a^*(a) = 0$$

The calculation for $b^* \smile b^* = 0$ is similar.

(3) We have

$$(a^* \smile b^*)(U) = a^*(U|_{[0,1]})b^*(U|_{[1,2]}) = a^*(a)b^*(b) = 1$$
$$(a^* \smile b^*)(L) = a^*(L|_{[0,1]})b^*(L|_{[1,2]}) = a^*(b)b^*(a) = 0$$

Moreover $C = U - L$, so $(a^* \smile b^*)(C) = (a^* \smile b^*)(U) - (a^* \smile b^*(L)) = 1$. This gives $a^* \smile b^* = C^*$

(4) We have

$$(b^* \smile a^*)(U) = b^*(U|_{[0,1]})a^*(U|_{[1,2]}) = b^*(a)a^*(b) = 0$$
$$(b^* \smile a^*)(L) = b^*(L|_{[0,1]})a^*(L|_{[1,2]}) = b^*(b)a^*(a) = 1$$

So $(b^* \smile a^*)(C) = (b^* \smile a^*)(U) - (b^* \smile a^*(L)) = -1$.

□

2.2 Wedge of spaces

We can define another CW-complex with the same number of cells as the torus, namely $S^1 \vee S^1 \vee S^2$. The boundary maps of the cellular cochain complex will also be 0, so we get the same cohomology groups (those two spaces also have the same homology groups). However, their cohomologies will not be isomorphic as graded rings. This implies in particular that the spaces are not homotopy equivalent, which we already knew from calculating π_1.

To describe the graded ring structure of a wedge, we need to define the product of graded rings.

Definition 2.5. Given graded rings E^*, F^*, define their product $E^* \times F^*$ as the graded ring with multiplication given by

$$(e_1, f_1)(e_2, f_2) = (e_1 e_2, f_1 f_2)$$

This is the categorical product in grRing.

The inclusions $i\colon X \to X \vee Y$ and $j\colon Y \to X \vee Y$ induce maps of graded rings $i^*\colon \mathrm{H}^*(X \vee Y; R) \to \mathrm{H}^*(X; R)$ and $j^*\colon \mathrm{H}^*(X \vee Y; R) \to \mathrm{H}^*(X; R)$. By the universal property of the product, we get a map of graded rings

$$(i^*, j^*)\colon \mathrm{H}^*(X \vee Y; R) \to \mathrm{H}^*(X; R) \times \mathrm{H}^*(Y; R).$$

Moreover, this map is an isomorphism in degrees ≥ 1.

Corollary 2.6. *Suppose $x, y \in \mathrm{H}^1(S^1 \vee S^1 \vee S^2)$, then $x \smile y = 0$. In particular,* $\mathrm{H}^*(S^1 \vee S^1 \vee S^2) \not\simeq \mathrm{H}^*(\mathbb{T})$.

Proof. We have $\mathrm{H}^1(S^1 \vee S^1 \vee S^2) \simeq \mathrm{H}^1(S^1 \vee S^1) \times \widetilde{\mathrm{H}}^1(S^2)$, but $\mathrm{H}^1(S^2) = 0$, so it is enough to prove for $x, y \in H^1(S^1 \vee S^1)$. However, $x \smile y \in \mathrm{H}^2(S^1 \vee S^1) = 0$. □