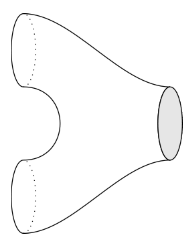
The Multiplicative Structure of Higher Bordism Categories

Leor Neuhauser

February 4, 2025

1 Background

Definition 1.1. Given two smooth closed n-manifolds M, N, a *bordism* between them is an $n+1$-manifold with boundary W such that $\partial W = M \sqcup N$.



This talk will start with a brief overview of the deep connection between bordisms and homotopy theory. We will then categorify both sides, and present higher bordisms categories and their connection to higher category theory. Along the way I will share some work in progress joint with Lior Yanovski and Shai Keidar.

Bordisms define an equivalence relation on n-manifolds, where two manifolds are equivalent if there exists a bordism connecting them. A classical result in differential topology is the classification of manifolds up to bordism. Let Ω_n be the set of equivalence classes of n-manifolds up to bordisms, then Ω_* has the structure of a graded ring, with addition given by disjoint union of manifolds and multiplication given by product of manifold.

Theorem 1.2 (Thom)**.** There is an isomorphism of graded rings

$$\Omega_* \simeq \mathbb{Z}/2[x_n \mid n \neq 2^t - 1],\ |x_n| = n$$

Consider now manifolds with a (stable) framing, which is a trivialization of the (stable) tangent bundle. We would like to classify the graded ring of framed manifolds up to framed bordisms, Ω_n^{fr}. Amazingly, this effort is equivalent to the classification of the stable homotopy groups of spheres.

Theorem 1.3 (Thom-Pontrjagin)**.** There is an isomorphism of graded rings

$$\Omega_*^{\text{fr}} \simeq \pi_*(\mathbb{S}).$$

2 Higher bordism categories

A basic tenet of category theory is that when we say two things are the same, we want to remember the data relating them. In our case, instead of looking only at equivalence classes of framed manifolds, we can consider the category whose objects are framed manifolds and whose morphisms are framed bordisms. In fact, we will consider a higher version of that, where we also consider bordisms-between-bordisms and so on.

From now on, all manifolds will be suitably framed, and we will not emphasize the details of the framings.

Definition 2.1. $\mathrm{Bord}_n^{\mathrm{fr}}$ is an (∞, n)-category, i.e. there are morphisms in every level but they are invertible above n, whose:

- Objects are closed 0-manifolds.
- 1-morphisms are bordisms, namely 1-manifolds with boundaries.
- 2-morphisms are bordisms-between-bordisms, namely 2-manifolds with corners.
- Etc. up to level n, where we have n-manifolds with higher corners.
- $n+1$-morphisms are given by diffeomorphisms, above them there are homotopies, and so on.

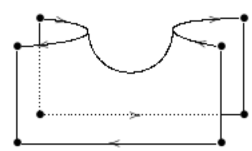


Figure 1: A 2-morphism in $\mathrm{Bord}_n^{\mathrm{fr}}$, image from Baez-Dolan

The additive structure of Ω_*^{fr} is easily categorified to a symmetric monoidal structure on $\mathrm{Bord}_n^{\mathrm{fr}}$, which is simply given by disjoint union of manifolds (with corners). However, categorifying the multiplicative structure of Ω_*^{fr} is more subtle. We would like the multiplication of manifolds to define a functor

$$\mathrm{Bord}_n^{\mathrm{fr}} \times \mathrm{Bord}_k^{\mathrm{fr}} \to \mathrm{Bord}_{n+k}^{\mathrm{fr}},$$

but such construction is impossible. To see why, note that the categorical product of two arrows is a commuting square:

$$\Delta^1 \times \Delta^1 = \begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ \downarrow & \Swarrow & \downarrow \\ \bullet & \longrightarrow & \bullet \end{array}$$

If we consider a pair of 1-morphisms from $\mathrm{Bord}_n^{\mathrm{fr}}$ and $\mathrm{Bord}_k^{\mathrm{fr}}$, namely 1-manifolds with boundary, then their multiplication will be a 2-manifold with corners. While this produces a 2-morphism in $\mathrm{Bord}_{n+k}^{\mathrm{fr}}$, it will in general not be invertible.

Instead of taking the regular categorical product, our situation requires a lax version where the commuting square above is replaced with a lax commuting square. The relevant construction is called the *Gray product*, denoted

$$\vec{\times} \colon \mathrm{Cat}_{(\infty,n)} \times \mathrm{Cat}_{(\infty,k)} \to \mathrm{Cat}_{(\infty,n+k)},$$

and it indeed satisfies

$$\Delta^1 \vec{\times} \Delta^1 = \begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ \big\downarrow & \Swarrow & \big\downarrow \\ \bullet & \longrightarrow & \bullet \end{array}$$

Similarly, multiplying more copies of Δ^1 produces higher lax cubes.

Theorem 2.2 (Keidar,Yanovski,N - WIP)**.** Multiplication of manifolds defines a functor

$$\mathrm{Bord}_n^{\mathrm{fr}} \vec{\times} \mathrm{Bord}_k^{\mathrm{fr}} \to \mathrm{Bord}_{n+k}^{\mathrm{fr}}.$$

Moreover, this functor is bilinear with respect to the additive structure, so it defines a symmetric monoidal functor

$$\mathrm{Bord}_n^{\mathrm{fr}} \vec{\otimes} \mathrm{Bord}_k^{\mathrm{fr}} \to \mathrm{Bord}_{n+k}^{\mathrm{fr}}.$$

3 Full dualizability

Similarly to how the framed bordism ring Ω_*^{fr} turned out to be a fundamental object in homotopy theory, the cobordism hypothesis posits that $\mathrm{Bord}_n^{\mathrm{fr}}$ has a fundamental role in higher category theory, as the free (∞, n)-category on a fully dualizable object. Let us first recall the definition of regular dualizability.

Definition 3.1. Let $\mathscr{C}$ be a symmetric monoidal $(\infty, 1)$-category. An object $X \in \mathscr{C}$ is called *dualizable* if there exists:

(1) Another object $X^\vee \in \mathscr{C}$,

(2) evaluation $\mathrm{ev} \colon X^\vee \otimes X \to \mathbb{1}$ and coevalutaion $\mathrm{coev} \colon \mathbb{1} \to X \otimes X^\vee$ morphisms, and

(3) zigzag identities

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{coev}\otimes\mathrm{id}} & X \otimes X^\vee \otimes X \\ & \diagdown\!\!\diagdown & \big\downarrow{\scriptstyle \mathrm{id}\otimes\mathrm{ev}} \\ & & X \end{array} \qquad \begin{array}{ccc} X^\vee & \xrightarrow{\mathrm{id}\otimes\mathrm{coev}} & X^\vee \otimes X \otimes X^\vee \\ & \diagdown\!\!\diagdown & \big\downarrow{\scriptstyle \mathrm{ev}\otimes\mathrm{id}} \\ & & X^\vee. \end{array}$$

For example, in Vect_k the dualizable objects are the finite dimensional vector spaces.

Definition 3.2. Let $\mathscr{C}$ be a symmetric monoidal (∞, n)-category. An object $X \in \mathscr{C}$ is called *n-fully dualizable* if:

- X is dualizable.

- The evaluation and coevaluation morphisms of X's duality have left and right adjoints.
- The unit and counit 2-morphisms of the above adjunctions have themselves left and right adjoints,
- and so on, up to the $n-1$-morphisms (note that the n-morphisms can't have adjoints without being invertible).

The object $\mathrm{pt} \in \mathrm{Bord}_n^{\mathrm{fr}}$ is n-fully dualizable. For example, to see that it is dualizable consider the 1-morphisms

with the arrow indicating the framing. The zigzag identities follow from the diffeomorphisms

Conjecture 3.3 (Baez-Dolan, Lurie)**.** $\mathrm{Bord}_n^{\mathrm{fr}}$ is the free symmetric monoidal (∞, n)-category with an n-fully dualizable object. That is, for any symmetric monoidal (∞, n)-category $\mathscr{C}$, a symmetric monoidal functor

$$F \colon \mathrm{Bord}_n^{\mathrm{fr}} \to \mathscr{C}$$

is determined by the choice of a single n-fully dualizable object

$$F(\mathrm{pt}) \in \mathscr{C}.$$

The cobordism hypothesis was proven for $n = 1$ by Harpaz, and it remains open for $n > 1$.

Notice that the cobordism hypothesis only takes the additive structure of $\mathrm{Bord}_n^{\mathrm{fr}}$ into account. Now that we also have a multiplicative structure, it is natural to ask how it relates to this conjectured universal property.

Theorem 3.4 (Keidar,Yanovski,N - WIP)**.** Given the cobordism hypothesis for some $n \geq 1$, the cobordism hypothesis for $n + 1$ is equivalent to the statement that the multiplication map

$$\mathrm{Bord}_n^{\mathrm{fr}} \vec{\otimes} \mathrm{Bord}_1^{\mathrm{fr}} \to \mathrm{Bord}_{n+1}^{\mathrm{fr}}$$

is an isomorphism.

Assuming the cobordism hypothesis, it will follow that the multiplication map

$$\mathrm{Bord}_n^{\mathrm{fr}} \vec{\otimes} \mathrm{Bord}_k^{\mathrm{fr}} \to \mathrm{Bord}_{n+k}^{\mathrm{fr}}$$

is an isomorphism for all $n, k \geq 1$. Taking the colimit, this implies that

$$\mathrm{Bord}_\infty^{\mathrm{fr}} \vec{\otimes} \mathrm{Bord}_\infty^{\mathrm{fr}} \to \mathrm{Bord}_\infty^{\mathrm{fr}}$$

is an isomorphism, making $\mathrm{Bord}_\infty^{\mathrm{fr}}$ into an idempotent algebra.