Rigid algebras in $(\infty, 2)$-categories

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1 Background

1.1 Duality

Let $(\mathscr{C}, \otimes, \mathbb{1})$ be a symmetric monoidal category, where by *category* we mean $(\infty, 1)$-category.

Definition 1.1. An object $X \in \mathscr{C}$ is called *dualizable* if there exist:

(1) a dual $X^\vee \in \mathscr{C}$,

(2) evaluation $\mathrm{ev}\colon X^\vee \otimes X \to \mathbb{1}$ and coevaluation $\mathrm{coev}\colon \mathbb{1} \to X \otimes X^\vee$ maps, and

(3) zigzag identities

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{coev}\otimes\mathrm{id}} & X \otimes X^\vee \otimes X \\ & \searrow^{=} & \downarrow{\scriptstyle \mathrm{id}\otimes\mathrm{ev}} \\ & & X \end{array} \qquad \begin{array}{ccc} X^\vee & \xrightarrow{\mathrm{id}\otimes\mathrm{coev}} & X^\vee \otimes X \otimes X^\vee \\ & \searrow^{=} & \downarrow{\scriptstyle \mathrm{ev}\otimes\mathrm{id}} \\ & & X^\vee. \end{array}$$

If a dual exists then it is essentially unique, and all the above data is contractible.

Example 1.2. A vector space is dualizable if and only if it is finite dimensional, with dual given by the dual vector space $V^\vee = \hom_k(V, k)$. Suppose V has a finite basis $\{e_1, \ldots, e_n\}$, the evaluation

$$\mathrm{ev}\colon V^\vee \otimes_k V \to k$$

is given by literal evaluation, and the coevaluation

$$\mathrm{coev}\colon k \to V \otimes_k V^\vee$$

chooses the element $\sum_{i=1}^n e_i \otimes e_i^* \in V \otimes V^\vee$ (the identity matrix).

Example 1.3. $R \in \mathrm{CAlg}(\mathrm{Sp})$ is a commutative ring spectrum, then the dualizable objects in $\mathrm{Mod}_R(\mathrm{Sp})$ are the compact modules.

Suppose $X, Y \in \mathscr{C}$ are dualizable and let $f\colon X \to Y$. Taking duals produces a map in the opposite direction, called the *transpose* $f^t\colon Y^\vee \to X^\vee$. Explicitly, the transpose is the composition

$$Y^\vee \xrightarrow{\mathrm{coev}_X} Y^\vee \otimes X \otimes X^\vee \xrightarrow{f} Y^\vee \otimes Y \otimes X^\vee \xrightarrow{\mathrm{ev}_Y} X^\vee$$

1.2 Frobenius algebras

Consider the vector space of $n \times n$-matrices $M_n(k)$, which is a k-algebra via multiplication of matrices. $M_n(k)$ is self dual, and the evaluation map is the trace of the multiplication:

$$M_n(k) \otimes_k M_n(k) \to M_n(k) \xrightarrow{\mathrm{Tr}} k.$$

Such self dual algebras are called *Frobenius algebras.*

Definition 1.4. An algebra $A \in \mathrm{Alg}(\mathscr{C})$ together with a map $\epsilon \colon A \to \mathbb{1}$ (not of algebras) is called a Frobenius algebra if

$$A \otimes A \xrightarrow{\mu} A \xrightarrow{\epsilon} \mathbb{1}$$

is an evaluation exhibiting A as self dual.

A Frobenius algebra is not just self dual, it is A-*linearly* self dual. To explain what this means, we introduce duality of bimodules.

Let $A, B, C \in \mathrm{Alg}(\mathscr{C})$ be algebras, and suppose $M \in {}_A\mathrm{BMod}_B(\mathscr{C})$ and $N \in {}_B\mathrm{BMod}_C(\mathscr{C})$. Assuming $\mathscr{C}$ has simplicial colimits that commute with the tensor product, we can construct the relative tensor product

$$M \otimes_B N \in {}_A\mathrm{BMod}_C(\mathscr{C})$$

as the colimit of the bar construction

$$M \otimes_B N = \varinjlim \left(\cdots \;\; M \otimes B \otimes B \otimes N \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; M \otimes B \otimes N \;\substack{\longrightarrow\\ \longrightarrow}\; M \otimes N \;\right)$$

Definition 1.5. A bimodule $M \in {}_A\mathrm{BMod}_B(\mathscr{C})$ is dualizable if there exist:

(1) a dual $N \in {}_B\mathrm{BMod}_A(\mathscr{C})$,

(2) an evaluation map $\mathrm{ev} \colon N \otimes_A M \to B$ in ${}_B\mathrm{BMod}_B(\mathscr{C})$,

(3) a coevaluation map $\mathrm{coev} \colon A \to M \otimes_B N$ in ${}_A\mathrm{BMod}_A(\mathscr{C})$, and

(4) zigzag identities.

Denote $\mathrm{LMod}_A(\mathscr{C}) = {}_A\mathrm{BMod}_{\mathbb{1}}(\mathscr{C})$ and $\mathrm{RMod}_A(\mathscr{C}) = {}_{\mathbb{1}}\mathrm{BMod}_A(\mathscr{C})$. We can view A both as a left module and as a right module over itself.

Proposition 1.6 ([Lur]). *Let $A \in \mathrm{Alg}(\mathscr{C})$ be an algebra with a map $\epsilon \colon A \to \mathbb{1}$. The following are equivalent:*

(1) *(A, ϵ) is a Frobenius algebra.*

(2) *Considering ϵ as a map $A \otimes_A A \to \mathbb{1}$, it is an evaluation exhibiting $A \in \mathrm{RMod}_A(\mathscr{C})$ as dual to $A \in \mathrm{LMod}_A(\mathscr{C})$.*

(3) *There exist a map $\delta \colon A \to A \otimes A$ in ${}_A\mathrm{BMod}_A(\mathscr{C})$ with isomorphisms*

$$(\epsilon \otimes \mathrm{id}_A)\delta \simeq \mathrm{id}_A \simeq (\mathrm{id}_A \otimes \epsilon)\delta$$

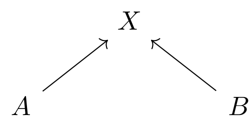
Remark 1.7. (1) A Frobenius algebra has a coalgebra structure with counit $\epsilon \colon A \to \mathbb{1}$ and comultiplication $\delta \colon A \to A \otimes A$. The above isomorphisms are the counitality relations.

(2) For conditions (1) and (3) we did not use relative tensor product. This equivalence holds even if $\mathscr{C}$ does not have simplicial colimits.

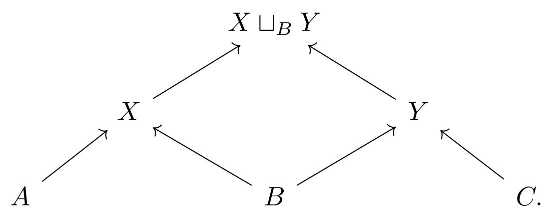
1.3 Cospans

In important example of Frobenius algebras will come from cospans. Let $\mathscr{C}$ be a category with finite colimits. The category $\mathrm{coSpan}(\mathscr{C})$ has:

- Objects the same as in $\mathscr{C}$.
- Morphisms from A to B are cospans:



  with composition given by pushouts



- The higher morphisms are equivalences of cospans.

There is a symmetric monoidal structure on $\mathrm{coSpan}(\mathscr{C})$ given by $A \sqcup B$, with unit $\emptyset$, which are not the coproduct and initial objects in $\mathrm{coSpan}(\mathscr{C})$. The map $\mathscr{C} \to \mathrm{coSpan}(\mathscr{C})$ which is identity on objects and sends a map $f \colon A \to B$ to the right way map $A \to B = B$ is symmetric monoidal, with $\mathscr{C}$ given the cocartesian structure. In particular this maps factors through commutative algebras:

$$\mathscr{C} \simeq \mathrm{CAlg}(\mathscr{C}^{\sqcup}) \to \mathrm{CAlg}(\mathrm{coSpan}(\mathscr{C})).$$

This endows every $A \in \mathrm{coSpan}(\mathscr{C})$ with a canonical commutative algebra structure, with multiplication $A \sqcup A \to A = A$.

Proposition 1.8. *The canonical commutative algebra structure on $A \in \mathrm{CAlg}(\mathcal{U})$ is canonically a Frobenius algebra via the counit $A = A \leftarrow \emptyset$*

Proof. The composition of multiplication and counit is $A \sqcup A \to A \leftarrow \emptyset$. The corresponding coevaluation is the mirror image $\emptyset \to A \leftarrow A \sqcup A$, the zigzag identities can be checked. □

Corollary 1.9. *For every cospan $A \to X \leftarrow B$, the transpose in* $\mathrm{coSpan}(\mathscr{C})$ *is the mirror image $B \to X \leftarrow A$.*

1.4 Rigid categories

Definition 1.10 (version 1)**.** A small symmetric monoidal category $\mathscr{D}$ is called *rigid* if every $X \in \mathscr{D}$ is dualizable.

Example 1.11. For $R \in \mathrm{CAlg}(\mathrm{Sp})$, the category of compact modules $\mathrm{Mod}_R(\mathrm{Sp})^\omega$ is rigid.

Let $\mathscr{C}$ be a presentably symmetric monoidal, stable category $\mathscr{C} \in \mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}})$, and assume that $\mathscr{C}$ is compactly generated. Being dualizable, like being compact, is a "smallness" condition, hence it is unreasonable to expect every object in $\mathscr{C}$ to be dualizable. However, it is resonable to expect every compact object to be dualizable.

Definition 1.12 (version 2)**.** A compactly generated $\mathscr{C} \in \mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}})$ is called *rigid* if $\mathscr{C}^\omega$ is a small rigid category, or equivalently if the dualizable objects and compact objects coincide.

Example 1.13. $\mathrm{Mod}_R(\mathrm{Sp})$ is rigid.

We would like to extend the definition of rigid to $\mathscr{C}$ which is not compactly generated. The naive generalization, in terms of compact objects, is ill-behaved in this case. Instead, we would like to reformulate this condition without referencing objects.

Lemma 1.14. *A compactly generated $\mathscr{C} \in \mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}})$ is rigid if and only if the following hold:*

(1) *The unit map $\mathrm{Sp} \to \mathscr{C}$ is internally left adjoint.*

(2) *The multiplication map $\mathscr{C} \otimes \mathscr{C} \to \mathscr{C}$ is internally left adjoint in ${}_A\mathrm{BMod}_A(\mathrm{Pr}^L_{\mathrm{st}})$, which means that it is internally left adjoint and a projection formula holds.*

Proof. To show that every dualizable object is compact, it is enough to show that the unit $\mathbb{1} \in \mathscr{C}$ is compact. The unit map

$$\mathbb{1} \otimes - : \mathrm{Sp} \to \mathscr{C}$$

always has a left adjoint given by the mapping spectrum

$$\mathrm{map}_{\mathscr{C}}(\mathbb{1}, -) : \mathscr{C} \to \mathrm{Sp}.$$

The unit map is internally left adjoint if and only if $\mathrm{map}_{\mathscr{C}}(\mathbb{1}, -)$ commutes with colimits, i.e. $\mathbb{1}$ is compact.

The other implication, that every compact is dualizable, follows from condition (2). □

Definition 1.15 (version 3, [GR19])**.** $\mathscr{C} \in \mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}})$ is rigid if it satisfies conditions (1) and (2) above.

Example 1.16. The following categories in $\mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}})$ are rigid:

(1) $\mathrm{Mod}_R(\mathrm{Sp})$.

(2) $\mathrm{Sh}(T; R)$ for T a compact Hausdorff topological space.

(3) The category of non-commutative motives (Efimov).

Given a rigid $\mathscr{C} \in \mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}})$, the right adjoints of the unit and multiplication are maps

$$\epsilon : \mathscr{C} \to \mathrm{Sp}$$
$$\delta : \mathscr{C} \to \mathscr{C} \otimes \mathscr{C} \in {}_A\mathrm{BMod}_A(\mathrm{Pr}^L_{\mathrm{st}})$$

which satisfy counitality, as a right adjoint to unitality.

Corollary 1.17. *A rigid* $\mathscr{C} \in \mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}})$ *is in particular a Frobenius algebra.*

Remark 1.18. While rigid categories are not necessarily compactly generated, they are dualizable, which in $\mathrm{Pr}^L_{\mathrm{st}}$ means they are compactly assembled. In fact, rigid categories can be viewed as a monoidal version of dualizable categories [Ram24].

2 Main talk

To define rigid presentably monoidal categories, we needed the symmetric monoidal structure of $\mathrm{Pr}^L_{\mathrm{st}}$ (for algebras) and the $(\infty, 2)$-categorical structure (for internal adjoints). We can consider the same definition in an any symmetric monoidal $(\infty, 2)$-category, which we will call a 2-category. Let us first present some 2-category theory.

2.1 Adjunctions

Let $\mathcal{U}$ be a 2-category. We can define adjoint morphisms in $\mathcal{U}$, using the usual definition with a unit and counit.

Suppose we have a commuting square in $\mathcal{U}$

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle k} \\ Z & \xrightarrow{g} & W \end{array}$$

where f and g have right adjoints f^R and g^R. The corresponding square with right adjoints does not neccesarily commute, but it always *lax* commutes

$$\begin{array}{ccc} Y & \xrightarrow{f^R} & X \\ {\scriptstyle k}\downarrow & \swarrow & \downarrow{\scriptstyle h} \\ W & \xrightarrow{g^R} & Z \end{array}$$

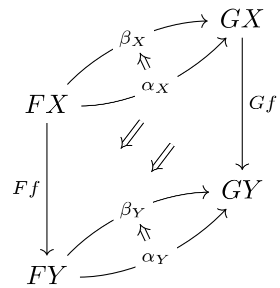
where the 2-morphism is called the *Beck-Chevalley map*. Let $F, G \colon \mathcal{U} \to \mathcal{V}$ be functors between 2-categories, and suppose $\alpha \colon F \to G$ is a natural transformation. If $\alpha_X : FX \to GX$ has a right adjoint $\alpha_X^R : GX \to FX$ for every $X \in \mathcal{V}$, then α_X^R assembles into a *lax natural transformation.*

Definition 2.1. Given 2-categorires $\mathcal{V}, \mathcal{U}$, there is a 2-category $\mathrm{Fun}_{\mathrm{lax}}(\mathcal{V}, \mathcal{V})$ whose:

- Objects are functors $F \colon \mathcal{U} \to \mathcal{V}$.
- Morphisms $\alpha \colon F \to G$ are lax natural transformation

$$\begin{array}{ccc} FX & \xrightarrow{\alpha_X} & GX \\ {\scriptstyle Ff}\downarrow & \swarrow & \downarrow{\scriptstyle Gf} \\ FY & \xrightarrow{\alpha_Y} & GY \end{array}$$

- 2-Morphisms $\alpha \to \beta$ are compatible 2-morphism



If a natural trnasformation $\alpha\colon F \to G$ has object wise right adjoints, then the lax natural transformation $\alpha^R\colon G \to F$ given by the Beck-Chevalley maps is right adjoint to α in $\mathrm{Fun}_{\mathrm{lax}}(\mathcal{U}, \mathcal{V})$.

Proposition 2.2 ([AGH24])**.** *Let $F, G\colon \mathcal{U} \to \mathcal{V}$ be functors of 2-categories.*

(1) *The left adjoints in* $\mathrm{Fun}_{\mathrm{lax}}(\mathcal{U}, \mathcal{V})$ *are the strong natural transformation that are object-wise left adjoint.*

(2) *The left adjoints in* $\mathrm{Fun}(\mathcal{U}, \mathcal{V})$ *are the natural transformation that are object-wise left adjoint such that the Beck-Chevalley maps are isomorphism.*

Now assume that $\mathcal{U}, \mathcal{V}$ are symmetric monoidal 2-categories. We will denote by $\mathrm{Fun}^{\otimes}_{\mathrm{lax}}(\mathcal{U}, \mathcal{V})$ the 2-category of *strongly* symmetric monoidal functors with *lax* natural transformations.

Given an operad $\mathscr{O}$, we want to consider algebras over $\mathscr{O}$ in $\mathcal{U}$ with lax $\mathscr{O}$-algebra maps. Denote by $\mathrm{Env}(\mathscr{O})$ the symmetric monoidal envelope of $\mathscr{O}$.

Definition 2.3. Given a symmetric monoidal 2-category $\mathcal{U}$ and an operad $\mathscr{O}$, define

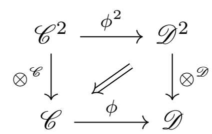
$$\mathrm{Alg}_{\mathscr{O}}(\mathcal{U}) := \mathrm{Fun}^{\otimes}(\mathrm{Env}(\mathscr{O}), \mathcal{U}).$$

$$\mathrm{Alg}^{\mathrm{lax}}_{\mathscr{O}}(\mathcal{U}) := \mathrm{Fun}^{\otimes}_{\mathrm{lax}}(\mathrm{Env}(\mathscr{O}), \mathcal{U})$$

Example 2.4. Let CAT_1 denote the 2-category of small 1-categories, with the cartesian symmetric monoidal structure. The symmetric monoidal envelope of the commutative operad is $\mathrm{Fin}^{\sqcup}$, and so

$$\mathrm{Fun}^{\otimes}_{\mathrm{lax}}(\mathrm{Fin}^{\sqcup}, \mathrm{CAT}_1) \simeq \mathrm{CAT}_1^{\otimes-\mathrm{lax}}$$

are symmetric monoidal categories with lax monoidal functors. To see why, let $\mathscr{C}, \mathscr{D}\colon \mathrm{Fin}^{\sqcup} \to \mathrm{CAT}_1$ be symmetric monoidal functors and $\phi\colon \mathscr{C} \to \mathscr{D}$ a lax transformation. The map $[2] \to [1]$ induces a lax square



with encodes lax monoidality

$$\phi(X) \otimes^{\mathscr{D}} \phi(Y) \to \phi(X \otimes^{\mathscr{C}} Y).$$

A version of Proposition 2.2 holds in the symmetric monoidal cases. This reproduces the proof that the right adjoint of a symmetric monoidal functor is lax symmetric monoidal. It also implies a projection formula for adjunctions in modules.

Corollary 2.5 (Projection formula)**.** *Let $A \in \mathrm{Alg}(\mathcal{U})$ and $f\colon M \to N \in \mathrm{LMod}_A(\mathcal{U})$. If f has a right adjoint in $\mathcal{U}$, then f^R receive the structure of a lax A-module map. The adjunction is in $\mathrm{LMod}_A(\mathcal{U})$ if and only if this lax structure is strong.*

A similar formula holds for right modules and bimodules.

2.2 Rigid algebras

Let $\mathcal{U}$ be a symmetric monoidal 2-category.

Definition 2.6. An algebra $A \in \mathrm{Alg}(\mathcal{U})$ is called *rigid* if

(1) The unit $\eta\colon \mathbb{1} \to A$ has a right adjoint $\epsilon\colon A \to \mathbb{1}$.

(2) The multiplication $\mu\colon A \otimes A \to A$ has a right adjoint $\delta\colon A \to A \otimes A$ in ${}_A\mathrm{BMod}_A(\mathcal{U})$, i.e. satisfying the projection formula.

Corollary 2.7. *A rigid $A \in \mathrm{Alg}(\mathcal{U})$ is in particular a Frobenius algebra, with counit ϵ and comultiplication δ.*

Rigid algebras can be thought of as 2-categorical improvements of Frobenius algebras, where we use adjunctions to get canonical candidates for the Frobenius algebra structure.

Consider an algebra map $f\colon A \to B \in \mathrm{Alg}(\mathcal{U})$. Equivalently, f is an algebra in the category of arrows $f \in \mathrm{Alg}(\mathrm{Fun}(\Delta^1, \mathcal{U}))$.

Proposition 2.8. *Let $A, B \in \mathrm{Alg}(\mathcal{U})$ be rigid and suppose $f\colon A \to B$ is an algebra map. Then f is rigid in $\mathrm{Alg}(\mathrm{Fun}_{\mathrm{lax}}(\Delta^1, \mathcal{U}))$.*

Proof. The unit and multiplication of f are given by

$$\begin{tikzcd} \mathbb{1} \arrow[r, "\eta_A"] \arrow[d, "\mathrm{id}_{\mathbb{1}}"'] & A \arrow[d, "f"] \\ \mathbb{1} \arrow[r, "\eta_B"] & B \end{tikzcd} \qquad \begin{tikzcd} A \otimes A \arrow[r, "\mu_A"] \arrow[d, "f \otimes f"'] & A \arrow[d, "f"] \\ B \otimes B \arrow[r, "\mu_B"] & B. \end{tikzcd}$$

Those square are left adjoint in $\mathrm{Fun}_{\mathrm{lax}}(\Delta^1, \mathcal{U})$ by Proposition 2.2, with right adjoints given by

$$\begin{tikzcd} A \arrow[r, "\epsilon_A"] \arrow[d, "f"'] & \mathbb{1} \arrow[d, "\mathrm{id}_{\mathbb{1}}"] \arrow[dl, Rightarrow, "\epsilon_f"'] \\ B \arrow[r, "\epsilon_B"] & \mathbb{1} \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\delta_A"] \arrow[d, "f"'] & A \otimes A \arrow[d, "f \otimes f"] \arrow[dl, Rightarrow, "\delta_f"'] \\ B. \arrow[r, "\delta_B"] & B \otimes B \end{tikzcd}$$

The projection formula for δ_f then follows from the projection formulas for δ_A and δ_B. ☐

In patricular, a map of rigid algebras is dualizable in $\mathrm{Fun}_{\mathrm{lax}}(\Delta^1, \mathcal{U})$. Here we see how the symmetric monoidal notion of duality mixes with the 2-categorical notion of adjunction:

Lemma 2.9 ([HSS17])**.** *An arrow* $f\colon X \to Y \in \mathcal{U}$ *is dualizable in* $\mathrm{Fun}_{\mathrm{lax}}(\Delta^1,\mathcal{U})$ *if and only if* X *and* Y *are dualizable in* $\mathcal{U}$ *and* f *has a right adjoint* f^R*, in which case the dual of* f *is*

$$f^\vee = (f^R)^t\colon X^\vee \to Y^\vee.$$

Corollary 2.10. *An algebra map between rigid algebra* $f\colon A \to B$ *is left adjoint to* f^t*. Moreover, the adjunction is in* $\mathrm{LMod}_A(\mathcal{U})$*.*

Proof. The first part follows from self duality $f \simeq (f^R)^t$. As f is a Frobenius algebra, the self duality should be self-linear, and in particular A-linear. This is verified with the projection formula. □

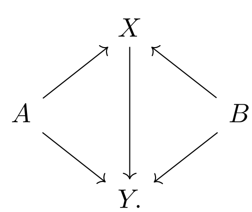
Remark 2.11. This property is known for monoidal maps between rigid categories, however even in this case our generalization provides something stronger, by allowing us say the maps are rigid.

Denote by $\mathrm{Alg}^{\mathrm{rig}}(\mathcal{U}) \subseteq \mathrm{Alg}(\mathcal{U})$ the full subcategory of rigid algebra. All 1-morphisms here are rigid, and in particular adjointable. Applying this argument recursively, we see that all 2-morphisms are rigid, and in particular adjointable. But in a 2-category this means that all 2-morphisms are invertible

Corollary 2.12. $\mathrm{Alg}^{\mathrm{rig}}(\mathcal{U})$ *is a 1-category.*

2.3 2-categorical cospans

Important examples of rigid algebras outside Pr^L come from 2-categorical cospans. Given a 1-category $\mathscr{C}$ with finite colimits, denote by $\mathrm{coSpan}_{1.5}(\mathscr{C})$ the 2-category whose objects and morphisms are as in $\mathrm{coSpan}(\mathscr{C})$, but whose 2-morphisms are maps between cospans:

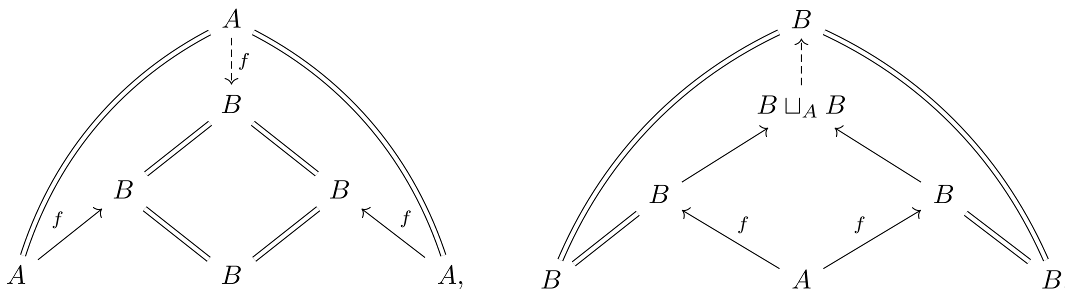


The notation $\mathrm{coSpan}_{1.5}(\mathscr{C})$ is to avoid confusion with $\mathrm{coSpan}_2(\mathscr{C})$, where the 2-morphisms are cospans-between-cospans.

We saw that every $A \in \mathrm{coSpan}(\mathscr{C})$ is canonically a Frobenius algebra. In $\mathrm{coSpan}_{1.5}(\mathscr{C})$, this structure comes from being a rigid algebra.

Lemma 2.13. *A cospan* $A \to X \leftarrow B$ *is left adjoint in* $\mathrm{coSpan}_{1.5}(\mathscr{C})$ *if and only if* $X \leftarrow B$ *is an isomorphism.*

Proof. We will present the "if" direction. Let $A \to B = B$ be a right way map, its right adjoint is the mirror image wrong way map $B = B \leftarrow A$, with unit and counit given by



∎

Corollary 2.14. *The canonical commutative algebra structure on* $A \in \mathrm{coSpan}(\mathscr{C})$ *is rigid.*

Proof. The unit and multiplication are given by right way map $\emptyset \to A = A$ and $A \sqcup A \to A = A$, and so they are left adjoints. It remains to check the projection formula. ∎

Every right way map $A \to B = B$ is a map of commutative algebras with the canonical structure, and rigidity implies that it is left adjoint to its transpose. And indeed, the transpose is the mirror image wrong way map.

We saw that every $A \in \mathscr{C}$ is a rigid algebra in $\mathrm{coSpan}_{1.5}(\mathscr{C})$, giving us a functor $\mathscr{C} \to \mathrm{CAlg}^{\mathrm{rig}}(\mathrm{coSpan}_{1.5}(\mathscr{C}))$. In fact, this classifies all rigid algebras there. To prove that, we will need the following lemma:

Lemma 2.15. *The functor* $\mathscr{C} \to \mathrm{CAlg}(\mathrm{coSpan}(\mathscr{C}))$ *is a fully-faithful left adjoint, with essential image those algebras where the unit is a right way map.*

Proposition 2.16. *The functor* $\mathscr{C} \to \mathrm{CAlg}^{\mathrm{rig}}(\mathrm{coSpan}_{1.5}(\mathscr{C}))$ *is an isomorphism.*

Proof. $\mathrm{CAlg}^{\mathrm{rig}}(\mathrm{coSpan}_{1.5}(\mathscr{C}))$ is a full subcategory of $\mathrm{CAlg}(\mathrm{coSpan}_{1.5}(\mathscr{C}))$, but it is also a 1-category, so it is a full subcategory of $\mathrm{CAlg}(\mathrm{coSpan}(\mathscr{C}))$. Thus, it is enough to check that every $A \in \mathrm{CAlg}^{\mathrm{rig}}(\mathrm{coSpan}_{1.5}(\mathscr{C}))$ belongs to the full subcategory $\mathscr{C} \hookrightarrow \mathrm{CAlg}(\mathrm{coSpan}(\mathscr{C}))$. Indeed, the unit of A is a left adjoint, and so it is a right way map. ∎

2.4 Universality of cospans

The category $\mathrm{coSpan}_{1.5}(\mathscr{C})$ has a universal property.

Proposition 2.17. *The right way map* $\mathscr{C} \to \mathrm{coSpan}_{1.5}(\mathscr{C})$ *is the initial symmetric monoidal functor from* $\mathscr{C}$ *into a 2-category satisfying the following:*

(1) *every morphism in* $\mathscr{C}$ *is sent to a left adjoint in* $\mathscr{C}$*, and*

(2) *for every pushout square,*

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & P \end{array}$$

the corresponding square in $\mathcal{U}$

$$\begin{array}{ccc} FA & \longrightarrow & FB \\ \downarrow & & \downarrow \\ FC & \longrightarrow & FP. \end{array}$$

satisfies the Beck-Chevalley condition.

That is, symmetric monoidal functors $\mathscr{C} \to \mathcal{U}$ satisfying the above correspond to symmetric monoidal functors $\mathrm{coSpan}_{1.5}(\mathscr{C}) \to \mathcal{U}$.

The universal property of cospans is deeply connected to rigid categories – both were studied in [GR19] to ultimately construct a six functors formalism on ind-coherent sheaves. We wish to formalize this connection as an adjunction $\mathrm{coSpan}_{1.5} \dashv \mathrm{CAlg}^{\mathrm{rig}}$, which holds up to some restriction. Before making this claim precise, let us understand the unit and counit of this adjunction:

(1) The unit is the isomorphism $\mathscr{C} \xrightarrow{\sim} \mathrm{CAlg}^{\mathrm{rig}}(\mathrm{coSpan}_{1.5}(\mathscr{C}))$.

(2) The counit $\mathrm{coSpan}_{1.5}(\mathrm{CAlg}^{\mathrm{rig}}(\mathcal{U})) \to \mathcal{U}$ will come from the forgetful functor $\mathrm{CAlg}^{\mathrm{rig}}(\mathcal{U}) \to \mathcal{U}$ via the universal property.

One problem with constructing the counit is that $\mathrm{coSpan}_{1.5}(\mathrm{CAlg}^{\mathrm{rig}}(\mathcal{U}))$ is not well defined unless $\mathrm{CAlg}^{\mathrm{rig}}(\mathcal{U})$ has pushouts. In $\mathrm{CAlg}(\mathcal{U})$, the pushout of $A \to B$ and $A \to C$ is given by the relative tensor product $B \otimes_A C$, which is the colimit of the bar construction.

Definition 2.18. Say that a symmetric monoidal 2-category $\mathcal{U}$ is *rigid bar compatible* if:

(1) The relative tensor product $B \otimes_A C$ exists, and is compatible with the tensor product, whenever A, B, C are rigid.

(2) $B \otimes_A C$ is itself rigid.

Denote by $\mathrm{Cat}_2^{\mathrm{RigBar}} \subseteq \mathrm{Cat}_2^{\otimes}$ the subcategory of rigid bar compatible 2-categories and symmetric monoidal functors that commute with the relevant relative tensor products.

Example 2.19. $\mathrm{Pr}^L_{\mathrm{st}}$ is rigid bar compatible [Ram24, AGK+22].

Lemma 2.20. *The functor* $\mathrm{coSpan}\colon \mathrm{Cat}_1^{\mathrm{rex}} \to \mathrm{Cat}_2^{\otimes}$ *factors as* $\mathrm{coSpan}\colon \mathrm{Cat}_1^{\mathrm{rex}} \to \mathrm{Cat}_2^{\mathrm{RigBar}}$.

Proof. Let $\mathscr{C} \in \mathrm{Cat}_1^{\mathrm{rex}}$. The relative tensor product in $\mathscr{C}$ is the pushout $B \sqcup_A C$, and the map $\mathscr{C} \to \mathrm{CAlg}(\mathrm{coSpan}(\mathscr{C}))$ commutes with colimits as a left adjoint. Thus, the relative tensor product of rigid algebras in $\mathrm{CAlg}(\mathrm{coSpan}_{1.5}(\mathscr{C}))$ is also $B \sqcup_A C$, which is itself rigid. Moreover, if a functor $\mathscr{C} \to \mathscr{D}$ commutes with pushouts it will commute with the relevant relative tensor products. ☐

On the other hand, the functor $\mathrm{CAlg}^{\mathrm{rig}}\colon \mathrm{Cat}_2^{\otimes} \to \mathrm{Cat}_1$ restricts to $\mathrm{CAlg}^{\mathrm{rig}}\colon \mathrm{Cat}_2^{\mathrm{RigBar}} \to \mathrm{Cat}_1^{\mathrm{rex}}$.

Theorem 2.21. There is an adjunction

$$\mathrm{coSpan}\colon \mathrm{Cat}_1^{\mathrm{rex}} \rightleftarrows \mathrm{Cat}_2^{\mathrm{RigBar}}\colon \mathrm{CAlg}^{\mathrm{rig}},$$

i.e. there is an isomorphism

$$\mathrm{Map}^{\mathrm{RigBar}}(\mathrm{coSpan}(\mathscr{C}), \mathcal{U}) \simeq \mathrm{Map}^{\mathrm{rex}}(\mathscr{C}, \mathrm{CAlg}^{\mathrm{rig}}(\mathcal{U})).$$

We already constructed the unit $\mathscr{C} \xrightarrow{\sim} \mathrm{CAlg}^{\mathrm{rig}}(\mathrm{coSpan}(\mathscr{C}))$, and for the counit we wish to apply the universal property on $\mathrm{CAlg}^{\mathrm{rig}}(\mathcal{U}) \to \mathcal{U}$. We know that every map of rigid algebra $A \to B$ is a left adjoint in $\mathcal{U}$, and we need to show that pushout squares satisfy the Beck-Chevalley property. This follows from the fact that the adjunction of rigid maps is A-linear.

Lemma 2.22. *Let $f\colon A \to B$ and $g\colon A \to C$ be maps of commutative algebras such that $B \otimes_A C$ exists, and consider the commutative square*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g} \\ C & \xrightarrow{f} & B \otimes_A C. \end{array}$$

If f is left adjoint in $\mathrm{Mod}_A(\mathcal{U})$, then this square satisfies Beck-Chevalley.

Note that the unit is an isomorphism, which implies that

$$\mathrm{Span}_{1.5}\colon \mathrm{Cat}_1^{\mathrm{rex}} \to \mathrm{Cat}_2^{\mathrm{RigBar}}$$

is fully faithful. In fact, this is true also when landing in the usual $\mathrm{Cat}_2^{\otimes}$.

Corollary 2.23. *The functor*

$$\mathrm{Span}_{1.5}\colon \mathrm{Cat}_1^{\mathrm{rex}} \to \mathrm{Cat}_2^{\otimes}$$

is fully faithful.

Proof. It is enough to show that every functor $F\colon \mathrm{coSpan}(\mathscr{C}) \to \mathrm{coSpan}(\mathscr{D})$ is rigid bar compatible, meaning that it commutes with the relative tensor products $B \sqcup_A C$. But composition of cospans is also given by pushouts, so F preserves them by virtue of being a functor. ☐

References

[AGH24] Fernando Abellán, Andrea Gagna, and Rune Haugseng. Straightening for lax transformations and adjunctions of $(\infty, 2)$-categories, 2024.

[AGK+22] D. Arinkin, D. Gaitsgory, D. Kazhdan, S. Raskin, N. Rozenblyum, and Y. Varshavsky. The stack of local systems with restricted variation and geometric langlands theory with nilpotent singular support, 2022.

[GR19] D. Gaitsgory and N. Rozenblyum. *A Study in Derived Algebraic Geometry: Volume I: Correspondences and Duality.* Mathematical Surveys and Monographs. American Mathematical Society, 2019.

[HSS17] Marc Hoyois, Sarah Scherotzke, and Nicoló Sibilla. Higher traces, noncommutative motives, and the categorified chern character. *Advances in Mathematics*, 309:97–154, 2017.

[Lur] Jacob Lurie. Higher algebra.

[Ram24] Maxime Ramzi. Locally rigid ∞-categories, 2024.