Cyclotomic Redshift

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1 Overview

This is a talk prepared for the Oberwolfach Arbeitsgemeinschaft about the disproof of the telescope conjecture. The talk is completely based on [BMCSY23].

The disproof of the telescope conjecture consists of two claims:

(1) every $K(n+1)$-local spectra is cyclotomically complete, and

(2) there exists some $T(n)$-local ring spectrum R such that $K_{T(n+1)}(R) := L_{T(n+1)}K(R)$ is not cyclotomically complete.

To approach (2), we need to understand how $T(n+1)$-local K-theory interacts with higher height cyclotomic extensions. Our goal in this talk is to prove that $T(n+1)$-local K-theory satisfies redshift:

Theorem A. Let R be a $T(n)$-local ring spectrum, then there is an isomorphism

$$K_{T(n+1)}(R[\omega_{p^\infty}^{(n)}]) \simeq K_{T(n+1)}(R)[\omega_{p^\infty}^{(n+1)}]$$

The main tool in proving this theorem will be descent for $T(n+1)$-local K-theory. In [CMNN22], Clausen, Mathew, Naumann and Noel proved descent along finite p-groups:

Theorem 1.1. Let $\mathscr{C}$ be an L_n^f-local category (i.e. a stable category with L_n^f-local mapping spectra) with G-action for a finite p-group G, then

$$K_{T(n+1)}(\mathscr{C}^{hG}) \xrightarrow{\sim} K_{T(n+1)}(\mathscr{C})^{hG},$$

$$K_{T(n+1)}(\mathscr{C}_{hG}) \xleftarrow{\sim} K_{T(n+1)}(\mathscr{C})_{hG}.$$

To consider arbitrary higher cyclotomic extensions, we need to generalize this result for groups in space. Namely, we will prove a version where G could be a π-finite p-group, meaning a group in spaces having finitely many non-vanishing homotopy groups, each a finite p-group.

Theorem B. Let $\mathscr{C}$ be an L_n^f-local category with G-action for a π-finite p-group G, then

$$K_{T(n+1)}(\mathscr{C}^{hG}) \xrightarrow{\sim} K_{T(n+1)}(\mathscr{C})^{hG},$$

$$K_{T(n+1)}(\mathscr{C}_{hG}) \xleftarrow{\sim} K_{T(n+1)}(\mathscr{C})_{hG}.$$

Note that because L_n^f is a smashing localization, the inclusion $\mathrm{Cat}_{L_n^f} \hookrightarrow \mathrm{Cat}_{\mathrm{perf}}$ is the forgetful $\mathrm{Mod}_{\mathrm{Perf}(L_n^f\mathbb{S})}(\mathrm{Cat}_{\mathrm{perf}}) \to \mathrm{Cat}_{\mathrm{perf}}$, so it preserves all limits and colimits. Thus, the limit and colimit $\mathscr{C}^{hG}$, $\mathscr{C}_{hG}$ can be taken either in $\mathrm{Cat}_{L_n^f}$ or in $\mathrm{Cat}_{\mathrm{perf}}$.

We will start by showing how Theorem A follows from Theorem B, and then prove Theorem B.

2 Cyclotomic redshift

Let R be a $T(n)$-local ring spectrum. We wish to apply Theorem B on $\mathrm{Perf}(R)$.

Lemma 2.1. *If R is an L_n^f-local ring spectrum then* $\mathrm{Perf}(R)$ *is an L_n^f-local category.*

Proof. As $R \in \mathrm{Mod}_{L_n^f\mathbb{S}}(\mathrm{Sp})$, and Perf is symmetric monoidal, it follows that

$$\mathrm{Perf}(R) \in \mathrm{Mod}_{\mathrm{Perf}(L_n^f\mathbb{S})}(\mathrm{Cat}_{\mathrm{perf}}) \simeq \mathrm{Cat}_{L_n^f}$$

$\square$

In fact, we only need Theorem B for the homotopy fixed points along a trivial G-action.

$$K_{T(n+1)}(\mathrm{Perf}(R)^{BG}) \xrightarrow{\sim} K_{T(n+1)}(\mathrm{Perf}(R))^{BG} = K_{T(n+1)}(R)^{BG} \simeq K_{T(n+1)}(R)[BG]$$

where the last isomorphism follows from the fact that $\mathrm{Sp}_{T(n+1)}$ is ∞-semiadditive. On the other hand, it follows from [CMNN22, Proposition 4.15] that

$$K_{T(n+1)}(\mathrm{Perf}(R)^{BG}) \simeq K_{T(n+1)}(\mathrm{Perf}(R[G])) = K_{T(n+1)}(R[G]).$$

So we conclude:

Corollary 2.2. *For every π-finite p-group G,*

$$K_{T(n+1)}(R[G]) \xrightarrow{\sim} K_{T(n+1)}(R)[BG]$$

Applying this to $G = B^n C_{p^r}$, we get

$$K_{T(n+1)}(R[B^n C_{p^r}]) \xrightarrow{\sim} K_{T(n+1)}(R)[B^{n+1} C_{p^r}].$$

Theorem 2.3 (Cyclotomic redshift)**.** Let $R \in \mathrm{CAlg}(\mathrm{Sp}_{T(n)})$, then for every $r \leq \infty$

$$K_{T(n+1)}(R[\omega_{p^r}^{(n)}]) \xrightarrow{\sim} K_{T(n+1)}(R)[\omega_{p^r}^{(n+1)}].$$

Proof. The case $r = \infty$ follows as a filtered colimit of $r < \infty$. For $r < \infty$, Recall that we defined $R[\omega_{p^r}^{(n)}]$ by the decomposition

$$R[B^n C_{p^r}] \simeq R[B^n C_{p^{r-1}}] \times R[\omega_{p^r}^{(n)}]$$

Moreover, the first term is compatible with the isomorphism of Corollary 2.2

$$\begin{tikzcd}
K_{T(n+1)}(R[B^n C_{p^r}]) \arrow[r, "\sim"] \arrow[d, "\sim"] & K_{T(n+1)}(R[B^n C_{p^{r-1}}]) \times K_{T(n+1)}(R[\omega_{p^r}^{(n)}]) \arrow[d, "\sim"] \\
K_{T(n+1)}(R)[B^{n+1} C_{p^r}] \arrow[r, "\sim"] & K_{T(n+1)}(R)[B^{n+1} C_{p^{r-1}}] \times K_{T(n+1)}(R)[\omega_{p^r}^{(n+1)}]
\end{tikzcd}$$

So by uniqueness of decompositions, there is also an isomorphism in the second term. There is more that needs to be said to show that this is an isomorphism of ring spectra, namely compatibility with the idempotent. $\square$

3 Descent

Let G be a π-finite p-group and $\mathscr{C} \in \mathrm{Cat}_{L_n^f}^{BG}$. To prove descent,

$$K_{T(n+1)}(\mathscr{C}^{hG}) \xrightarrow{\sim} K_{T(n+1)}(\mathscr{C})^{hG},$$

$$K_{T(n+1)}(\mathscr{C}_{hG}) \xleftarrow{\sim} K_{T(n+1)}(\mathscr{C})_{hG},$$

we will first make a few simplifying maneuvers. The homotopy fixed points/orbits are limits/colimits along BG, which is a connected π-finite p-space. We will prove this result more generally for (co)limits along any π-finite p-space A, not necessarily connected. In other words, we want to prove the functor

$$K_{T(n+1)} \colon \mathrm{Cat}_{L_n^f} \to \mathrm{Sp}_{T(n+1)}$$

commutes with π-finite p-space indexed (co)limits.

Definition 3.1. A category $\mathscr{C} \in \mathrm{Cat}_{\mathrm{perf}}$ is *n-monochromatic* if it is L_n^f-local and $L_{n-1}^f \mathscr{C} = 0$.

Consider the subcategory of n-monochromatic categories $\mathrm{Cat}_{M_n^f} \hookrightarrow \mathrm{Cat}_{L_n^f}$. This inclusion has a right adjoint $M_n^f \colon \mathrm{Cat}_{L_n^f} \to \mathrm{Cat}_{M_n^f}$, given by the fiber

$$M_n^f \mathscr{C} \to \mathscr{C} \to L_{n-1}^f \mathscr{C}.$$

Lemma 3.2. *The functor $M_n^f \colon \mathrm{Cat}_{L_n^f} \to \mathrm{Cat}_{M_n^f}$ commutes with all limits and colimits.*

Proof. M_n^f commutes with limits as a right adjoint. Composing with the (colimit preserving) left adjoint inclusion

$$\mathrm{Cat}_{L_n^f} \xrightarrow{M_n^f} \mathrm{Cat}_{M_n^f} \hookrightarrow \mathrm{Cat}_{L_n^f}$$

is given by tensoring with $M_n^f \operatorname{Perf}(L_n^f \mathbb{S})$, so it also preserves colimits. Thus, M_n^f preserves colimits. $\square$

Purity tells us that for $\mathscr{C} \in \mathrm{Cat}_{L_n^f}$, $K_{T(n+1)}(\mathscr{C})$ depends only on the n-th monochromatic part $M_n^f \mathscr{C}$, meaning that there is a commuting triangle

$$\begin{tikzcd} \mathrm{Cat}_{L_n^f} \arrow[r, "K_{T(n+1)}"] \arrow[d, "M_n^f"'] & \mathrm{Sp}_{T(n+1)} \\ \mathrm{Cat}_{M_n^f} \arrow[ur, "K_{T(n+1)}"'] & \end{tikzcd}$$

Using Lemma 3.2, we reduce our goal to proving that

$$K_{T(n+1)} \colon \mathrm{Cat}_{M_n^f} \to \mathrm{Sp}_{T(n+1)}$$

commutes with π-finite p-space indexed (co)limits. This seems just as complicated, but now we can use the following fact:

Proposition 3.3. $\mathrm{Cat}_{M_n^f}$ *is ∞-semiadditive*

Proof. The idea is that $\mathrm{Cat}_{M_n^f} \simeq \mathrm{Pr}^L_{T(n),\omega}$, which is the subcategory of $\mathrm{Pr}^L_{T(n)}$ on compactly generated categories and internal left adjoints. $\mathrm{Pr}^L_{T(n)}$ is ∞-semiadditive as a full subcategory of Pr^L that is closed under colimits, and one can show that the ambidexterity data restricts to $\mathrm{Pr}^L_{T(n),\omega}$. □

From the fact that both $\mathrm{Cat}_{M_n^f}$ and $\mathrm{Sp}_{T(n+1)}$ are ∞-semiadditive, we immediately see that commuting with π-finite p-spaces indexed limits and colimits is equivalent. In fact, even though we ultimately use limit preservation, it will be easier to prove the result for colimits. Moreover, using ∞-semiadditivity, it is enough to check only very specific kinds of colimits.

Proposition 3.4. *Let $F : \mathscr{C} \to \mathscr{D}$ be a functor between p-typically ∞-semiadditive categories. If F commutes with constant colimits indexed on π-finite p-spaces concentrated in a single homotopy degree, then it commutes with all π-finite p-space indexed (co)limits.*

Proof. The proof is by induction on the truncation level m of the indexing space A. The base case $m = -2$ is trivial, let $m \geq -1$. We will start by considering constant colimits over A. Denote $B = A_{\leq m-1}$ and $f : A \to B$ the canonical map, taking colimit along A is the same as taking the colimits along the fibers $A_b \to A \xrightarrow{f} B$ for every $b \in B$ and then taking colimit along B. However, each fiber is concentrated in homotopy degree m, and the diagram is constant on each fiber, so by assumption F commutes with the fiber-wise colimits. Moreover, B is $(m-1)$-truncated, so by the inductive assumption F commutes with B-indexed colimits.

We will now extend to non-constant (co)limits. Because F commutes with $(m-1)$-finite p-space indexed (co)limits, it follows from [CSY20b] that there is a commuting diagram relating the norm maps and the assembly maps for F:

$$\begin{tikzcd} FA_! \arrow[r, "\mathrm{Nm}_{\mathscr{C}}", "\sim"'] & FA_* \arrow[d, "\beta_*"] \\ A_!F \arrow[u, "\beta_!"] \arrow[r, "\mathrm{Nm}_{\mathscr{D}}", "\sim"'] & A_*F \end{tikzcd}$$

so it follows that $\beta_!$ admits a retract. On the other hand, using the wrong way adjunction $A^* \dashv A_!$ in $\mathscr{C}$, we have the following diagram:

$$\begin{tikzcd} & A_!FA^*A_! \arrow[r, "\epsilon"] \arrow[d, "\sim"'] & A_!F \arrow[d, "\beta_!"] \\ FA_! \arrow[r, "\eta"] \arrow[rr, bend right, equal] & FA_!A^*A_! \arrow[r, "\epsilon"] & FA_! \end{tikzcd}$$

where the middle isomorphism is the assembly along a constant diagram. This implies that $\beta_!$ admits a section, from which it follows that $\beta_!$ is an isomorphism, and so also β_*. $\square$

Theorem 3.5. The functor

$$K_{T(n+1)} \colon \mathrm{Cat}_{M_n^f} \to \mathrm{Sp}_{T(n+1)}$$

commutes with π-finite p-space indexed (co)limits.

Proof. By Proposition 3.4, it is enough to show for constant colimits concentrated in a single homotopical degree m. Namely, for such spaces A, we need to show that the assembley map

$$K_{T(n+1)}(\mathscr{C})[A] \to K_{T(n+1)}(\mathscr{C}[A])$$

is an isomorphism.

For $m = 0$, A is a finite set and $K_{T(n+1)}$ is exact, so in particular commutes with finite coproducts. For $m = 1$, $A = BG$ for G a finite p-group so the result follows from [CMNN22]. We proceed by induction on $m \geq 2$. By a corollary of the Schwede-Shipley Theorem [SS03], every $\mathscr{C} \in \mathrm{Cat}_{\mathrm{perf}}$ can be written as a filtered colimit

$$\mathscr{C} \simeq \operatorname{colim} \mathrm{Perf}(R_i)$$

where R_i is an endomorphism ring of an object in $\mathscr{C}$. As $\mathscr{C} \in \mathrm{Cat}_{M_n^f}$, it follows that $R_i \in M_n^f\mathrm{Sp}$, and as a full subcategory of $\mathscr{C}$ it follows that $\mathrm{Perf}(R_i) \in \mathrm{Cat}_{M_n^f}$. Thus, as $K_{T(n+1)}$ commutes with filtered colimits, it is enough to prove the case $\mathscr{C} = \mathrm{Perf}(R)$. Note that A is connected, so by semiadditivity $\mathrm{Perf}(R)[A] \simeq \mathrm{Perf}(R)^A \simeq \mathrm{Perf}(R[\Omega A])$, and the assembly map is thus

$$K_{T(n+1)}(R)[A] \to K_{T(n+1)}(R[\Omega A]).$$

Moreover, as $m \geq 2$, ΩA is also connected.

For height $n = 0$, $R[\Omega A] \simeq R$ and $K_{T(1)}(R)[A] \simeq K_{T(1)}(R)$ [CSY20a], and the assembly map is identified with the identity. Assume $n \geq 1$. Using the bar construction, we can write $A \simeq \operatorname{colim}_{\Delta^{\mathrm{op}}} A_k$ where $A_k = (\Omega A)^k$. Consider the commutative diagram

$$\begin{tikzcd} \operatorname{colim}_{\Delta^{\mathrm{op}}} K_{T(n+1)}(R)[A_k] \arrow[r] \arrow[d] & \operatorname{colim}_{\Delta^{\mathrm{op}}} K_{T(n+1)}(R[\Omega A_k]) \arrow[d] \\ K_{T(n+1)}(R)[\operatorname{colim}_{\Delta^{\mathrm{op}}} A_k] \arrow[r] & K_{T(n+1)}(R[\Omega \operatorname{colim}_{\Delta^{\mathrm{op}}} A_k]) \end{tikzcd}$$

We want to show that the bottom map is an isomorphism, we will show that all the others are:

- Top follows from the inductive hypothesis, as A_k is concentrated in homotopy degree $m-1$.
- Left is because $K_{T(n+1)}(R)[-]$ commutes with colimits
- Right is because Ω, $R[-]$ and $K_{T(n+1)}$ commutes with sifted colimits for $n \geq 1$.

$\square$

References

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