CYCLOTOMIC AND POLYGONIC SPECTRA

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1. REMINDER

To construct a witness against the telescope conjecture, we will use algebraic K-theory. $K : \mathrm{Cat}_{\mathrm{perf}} \to \mathrm{Sp}$ is the universal additive functor with a map from $\Sigma^\infty \mathcal{C}^{\simeq}$. Informally, $K(\mathcal{C}) \in \mathrm{Sp}$ is generated by $\mathcal{C}^{\simeq}$ such that for an exact sequence $X \to Y \to Z$, $[Y] = [X] + [Z]$.

For a commutative ring spectra R, $K(R) = K(\mathrm{Mod}_R(\mathrm{Sp}))$.

K-theory is very hard to calculate. A helpful approach is to approximate K with other localizing functors. We already saw THH,

Definition 1.1. $\mathrm{THH}(R)$ is the colimit of the cyclic diagram

$$\cdots \to R \otimes R \otimes R \to R \otimes R \rightrightarrows R$$

(there is a more general way to define THH on an idempotent complete category, as the dimension, but we will stick to the ring spectrum case)

There is a map $K(R) \to \mathrm{THH}(R)$, called the Denis trace map. But $\mathrm{THH}(R)$ has more structure. Each level of the cyclic bar construction has a C_n action, the resulting colimit has a $\mathbb{T} = S^1$ action (can be seen from cobordism using dim). Define $\mathrm{TC}^-(R) = \mathrm{THH}(R)^{h\mathbb{T}}$, there is a map $\mathrm{TC}^- \to \mathrm{THH}$ and $K \to \mathrm{THH}$ factors through it.

$\mathrm{THH}(R)$ has even further structure, for that we will need to define the Tate diagonal. Let p be a prime. Generally, there is no diagonal map $X \to X^{\otimes p}$ that is symmetric, in the sense that it factors through the fixed points $(X^{\otimes p})^{hC_p} \to X^{\otimes p}$. but we can define the *Tate diagonal* $\Delta_p : X \to (X^{\otimes p})^{tC_p}$. The functor $((-)^{\otimes p})^{tC_p}$ is exact, so it is enough to specify on $X = \mathbb{S}$,

$$\mathbb{S} \to \mathbb{S}^{hC_p} \xrightarrow{\sim} (\mathbb{S}^{\otimes p})^{hC_p} \to (\mathbb{S}^{\otimes p})^{tC_p}.$$

where $\mathbb{S}$ has the trivial C_p action. The second map comes from the fact that the equivalence $\mathbb{S} \xrightarrow{\sim} \mathbb{S}^{\otimes p}$ is C_p-equivariant.

$(-)^{tC_p}$ is lax symmetric monoidal, so it extends to a map of cyclic diagrams

$$\begin{array}{ccccccc}
 & & \overset{C_3}{\circlearrowleft} & & \overset{C_2}{\circlearrowleft} & & \\
\cdots & \longrightarrow & R^{\otimes 3} & \longrightarrow & R^{\otimes 2} & \longrightarrow & R \\
 & & \downarrow{\scriptstyle \Delta_p^{\otimes 3}} & & \downarrow{\scriptstyle \Delta_p^{\otimes 2}} & & \downarrow{\scriptstyle \Delta_p} \\
\cdots & \longrightarrow & (R^{\otimes 3p})^{tC_p} & \longrightarrow & (R^{\otimes 2p})^{tC_p} & \longrightarrow & (R^{\otimes p})^{tC_p} \\
 & & \underset{C_3}{\circlearrowright} & & \underset{C_2}{\circlearrowright} & &
\end{array}$$

Which realizes to a $\mathbb{T}$-equivariant *Frobenius map*

$$\varphi_p : \mathrm{THH}(R) \to \mathrm{THH}(R)^{tC_p}$$

where $\mathrm{THH}(R)^{tC_p}$ has the residual $\mathbb{T}/C_p \simeq \mathbb{T}$ action. Taking fixpoints we get maps

$$\varphi_p^{h\mathbb{T}} : \mathrm{THH}(R)^{h\mathbb{T}} \to (\mathrm{THH}(R)^{tC_p})^{h\mathbb{T}}$$

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which organize to

$$\prod_p \varphi_p^{h\mathbb{T}} : \mathrm{THH}(R)^{h\mathbb{T}} \to \prod_p (\mathrm{THH}(R)^{tC_p})^{h\mathbb{T}}$$

. For every X with $\mathbb{T}$-action there are canonical maps

$$X^{h\mathbb{T}} \simeq (X^{hC_p})^{h\mathbb{T}/C_p} \simeq (X^{hC_p})^{h\mathbb{T}} \to (X^{tC_p})^{h\mathbb{T}}$$

In particular for $X = \mathrm{THH}(R)$ we get the canonical map

$$can : \mathrm{THH}(R^{h\mathbb{T}}) \to \prod_p (\mathrm{THH}(R)^{tC_p})^{h\mathbb{T}}$$

Define

$$\mathrm{TC}(R) = \mathrm{Eq}(\prod_p \varphi_p^{h\mathbb{T}}, can) = \mathrm{Fib}(\prod_p \varphi_p^{h\mathbb{T}} - can)$$

TC is a "closer" approximation of K-theory, There is a cyclotomic trace map $K \to \mathrm{TC} \to \mathrm{TC}^- \to \mathrm{THH}$. It can be a good approximation, for example

Theorem 1.2 (Dundas-Goodwillie-McCarthy)**.** *If $R \to S$ is a map of connective ring spectra such that $\pi_0 R \to \pi_0 S$ has nilpotent kernel, then*

$$\begin{array}{ccc} K(R) & \longrightarrow & \mathrm{TC}(R) \\ \downarrow & & \downarrow \\ K(S) & \longrightarrow & \mathrm{TC}(S) \end{array}$$

is a pullback square.

Example 1.3.
- Quilen, algebraic K-theory:
$$K_n(\mathbb{F}_p) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n = 2i \\ \mathbb{Z}/(p^i - 1) & n = 2i - 1 \end{cases}$$
(also for q a p-power)
- Bokstedt:
$$\mathrm{THH}(\mathbb{F}_p) = \mathbb{F}_p[\sigma]$$
with σ of degree 2.
- $$\mathrm{TC}^-(\mathbb{F}_p) = \mathbb{Z}_p[\tilde{u}, v]/(\tilde{u}v - p)$$
with $\tilde{u}, v$ of degree 2, -2.
- $$\mathrm{TC}_n(\mathbb{F}_p) = \begin{cases} \mathbb{Z}_p & n = 0, -1 \\ 0 & else \end{cases}$$

2. Cyclotomic Spectra

We saw that THH has extra structure, and we defined TC as the invariants of this structure. We want a more conceptual way to say this. Consider the following example

Example 2.1. Consider the category GSet of sets with a G-action. It is symmetric monoidal with Cartesian product, and the unit is the point with the trivial G-action. Maps from the unit $\hom(1, X)$ gives precisely the fixed points of X

Our strategy is to define a category CycSp spectra with the extra structure that THH has, with a symmetric monoidal structure, such that maps from the unit will be the invariants of this structure, namely TC.

Definition 2.2 (Nikolaus, Scholze)**.** (1) A *cyclotomic spectrum* is a spectrum with $\mathbb{T}$-action $X \in \mathrm{Sp}^{B\mathbb{T}}$ and for every prime p a $\mathbb{T}$-equivariant Frobenius map $\varphi_p : X \to X^{tC_p}$.

(2) For a prime p, a *p-typical cyclotomic spectrum* is a spectrum with a C_{p^∞}-action $X \in Sp^{BC_{p^\infty}}$, where $C_{p^\infty = \mathbb{Q}_p/\mathbb{Z}_p}$ is the subgtoup of $\mathbb{T}$ of p-power torsion, and a C_{p^∞}-equivaruant map $\varphi_p : X \to X^{tC_p}$.

For our mission, we will mainly need the p-typical case when X is p-complete, in which case we can replace C_{p^∞} with $\mathbb{T}$. But today we will develop the general theory.

Example 2.3. (1) For every ring spectrum R, $\mathrm{THH}(R)$.

(2) $\mathbb{S}$ with the trivial $\mathbb{T}$-action and Frobenius maps $\mathbb{S} \to \mathbb{S}^{hC_p} \to \mathbb{S}^{tC_p}$. This is the same as $\mathrm{THH}(\mathbb{S})$, and we will also denote it by $\mathbb{S}$.

This assembles into a stable category CycSp, where maps between cyclotomic spectra are $\mathbb{T}$-equivariant maps such that the Frobenius maps commute, with s.m. structure given by $\otimes$ on the underlying spectra (using the fact that $(-)^{tC_p}$ is lax s.m.). The unit is given by $\mathbb{S}$.

Definition 2.4. For $X \in \mathrm{CycSp}$, $\mathrm{TC}(X) = \hom_{\mathrm{CycSp}}(\mathbb{S}, X)$

Proposition 2.5. $\mathrm{TC}(X)$ *is calculated as*

$$\mathrm{TC}(X) = \mathrm{Eq}(\prod_p \varphi_p^{h\mathbb{T}}, can)$$

In particular $\mathrm{TC}(\mathrm{THH}(R)) = \mathrm{TC}(R)$.

Why is this perspective important? We use TC as a good approximation of K. However, to calculate TC, it will be beneficial to do calculation inside CycSp, and only later take maps from the unit. This is similar to how we stay in the category of Sp as long as we can, and only at the end take π_*.

3. TR

We want to define a version of THH with coefficents,

Definition 3.1. Suppose R is a ring spectrum and M an R-bimodule. Consider the simplicial diagram

$$\cdots \to M \otimes R \otimes R \to M \otimes R \rightrightarrows M$$

$\mathrm{TR}(R, M)$ is the colimit of this diagram.

Example 3.2. $\mathrm{THH}(R, R) = \mathrm{THH}(R)$

THH with coefficients also has extra structure. For $n > 0$, we want to define a C_n action on $\mathrm{THH}(R, M^{\otimes_R n})$ We will start with $n = 2$. To do so, We will define a variant $\mathrm{THH}(R, M; R, M)$ For convenience we will add labels $\mathrm{THH}(R_0, M_0; R_1, M_1)$. It is defined as the colimit

$$\cdots \to M_0 \otimes M_1 \otimes R_0 \otimes R_1 \otimes R_0 \otimes R_1 \to M_0 \otimes M_1 \otimes R_0 \otimes R_1 \rightrightarrows M_0 \otimes M_1$$

where the first two maps are given by

$$\begin{aligned} m_0 \otimes m_1 \otimes r_0 \otimes r_1 &\mapsto m_0 r_1 \otimes m_1 r_0 \\ &\mapsto r_0 m_0 \otimes r_1 m_1. \end{aligned}$$

The other exterior maps are defined the same, and the interior maps are multiplication of $R_0 \otimes R_1$. Basically, think of $M_0 \otimes M_1$ as an $R_0 \otimes R_1$-module on the left and $R_1 \otimes R_0$-module on the right (drawing the circle can help). We have maps of simplicial diagrams

$$\begin{array}{ccc} & m_0 \otimes m_1 \otimes r_0 \otimes r_1 & m_0 r_1 \otimes m_1 r_0 \\ & & r_0 m_0 \otimes r_1 m_1 \\ \ldots \longrightarrow & M_0 \otimes M_1 \otimes R_0 \otimes R_1 \longrightarrow & M_0 \otimes M_1 \\ & \downarrow & \downarrow \\ \ldots \longrightarrow & M_0 \otimes_{R_1} M_1 \otimes R_0 \longrightarrow & M_0 \otimes_{R_1} M_1 \\ & m_0 r_1 \otimes_{R_1} m_1 \otimes r_0 & m_0 r_1 \otimes_{R_1} m_1 r_0 \\ & & r_0 m_0 \otimes_{R_1} r_1 m_1 \end{array}$$

which gives an equivalence

$$\mathrm{THH}(R_0, M_0; R_1, M_1) \xrightarrow{\sim} \mathrm{THH}(R_0, M_0 \otimes_{R_1} M_1).$$

It makes sense that this will turn into an equivalence, because taking the colimit of the upper diagram in particular coequalizes $m_0 r_1 \otimes m_1$ and $m_0 \otimes r_1 m_1$ Similarly, there is an equivalence

$$\mathrm{THH}(R_0, M_0; R_1, M_1) \xrightarrow{\sim} \mathrm{THH}(R_1, M_1 \otimes_{R_0} M_0).$$

There is a C_2 action on $\mathrm{THH}(R_0, M_0; R_1, M_1)$, switching 0 and 1, which induces a C_2 action on $\mathrm{THH}(R, M \otimes_R M)$. There is a similar story with a C_n-action on $\mathrm{THH}(R, M^{\otimes_R n})$, by constructing $\mathrm{THH}(R, M; \ldots; R, M)$, where $R^{\otimes n}$ acts on $M^{\otimes n}$ on the left and acts shifted on the right.

The Tate diagonal defines a map

$$\begin{array}{ccccc} \ldots & \longrightarrow & M^{\otimes n} \otimes R^{\otimes n} & \longrightarrow & M^{\otimes n} \\ & & \Big\downarrow \Delta_p & & \Big\downarrow \Delta_p \\ \ldots & \longrightarrow & (M^{\otimes np} \otimes R^{\otimes np})^{tC_p} & \longrightarrow & (M^{\otimes np})^{tC_p} \end{array}$$

using the fact that $(-)^{tC_p}$ is lax symmetric monoidal. This realizes to a C_n-equivariant maps $\varphi_{p,n} : \mathrm{THH}(R, M^{\otimes n}) \to \mathrm{THH}(R, M^{\otimes np})^{tC_p}$ Those maps assemble to

$$\prod_n (\mathrm{THH}(R, M^{\otimes n}))^{hC_n} \to \prod_p \prod_n (\mathrm{THH}(R, M^{\otimes np})^{tC_p})^{hC_n}$$

There is also a canonical map, coming from canonical maps $\mathrm{THH}(R, M^{\otimes np})^{hC_{np}} \to (\mathrm{THH}(R, M^{\otimes np})^{tC_p})^{hC_n}$. $\mathrm{TR}(R, M)$ is the equalizer of both those maps. Define also $\mathrm{TR}(R) = \mathrm{TR}(R, R)$.

4. Polygonic Spectra

Like in the case of TC, we will give a more conceptual definition of TR, as maps from the unit in an appropriate category.

Definition 4.1. A subset $T \subseteq \mathbb{N}_{>0}$ is a *truncation set* if for all $xy \in T$ both $x \in T$ and $y \in T$

Example 4.2.
(1) $\mathbb{N}_{>0}$
(2) $\{1\}$
(3) $\{1, \ldots, n\}$
(4) $\{1, p\}$

For a truncation set T we will denote $T/p = \{x \in T \mid xp \in T\}$. This is non-empty iff $p \in T$.

Definition 4.3 (Krause, McCandless, Nikolaus)**.** For T a truncation set, a T-*polygonic spectrum* is a series of spectra with C_n action $X \in \prod_{n \in T} \mathrm{Sp}^{BC_n}$, and for every prime p and $n \in T/p$ a C_n-equivariant map

$$\varphi_{p,n} : X_n \to X_{np}^{tC_p}.$$

If $T = \{1\}$, we simply get spectra. For our mission we will use $T = \{1, p\}$, which gives the structure of two spectra X_1, X_p and a map $X_1 \to X_p^{tC_p}$. For now we will develop generally $T = \mathbb{N}_{>0}$. An $\mathbb{N}_{>0}$-polygonic spectra will simply be called a polygonic spectra.

Example 4.4.
(1) For R a ring spectrum and M an R-bimodule, $\mathrm{THH}(R, M)$.
(2) In particular, $\mathrm{THH}(R)$
(3) the sphere spectrum $\mathbb{S}$ at every level, with trivial C_n-action, and $\varphi_{p,n} : \mathbb{S} \to \mathbb{S}^{tC_p}$. This is the same as $\mathrm{THH}(\mathbb{S})$, and will also be denoted by $\mathbb{S}$, or $\mathbb{S}_T$ when restricted to a truncation set T.

T-Polygonic spectra assemble into a stable category PgcSp_T, where maps between polygonic spectra being maps in $\prod_{n \in T} \mathrm{Sp}^{BC_n}$ that commute with $\varphi_{p,n}$, with s.m. structure coming from the underlying spectra. The unit is $\mathbb{S}_T$.

Definition 4.5. For $X \in \mathrm{PgcSp}_T$, define $\mathrm{TR}_T(X) = map_{\mathrm{PgcSp}}(\mathbb{S}_T, X)$.

Proposition 4.6. *For X a T-polygonic spectra,* $\mathrm{TR}_T(X)$ *is given by the equalizer of the canonical map and the map coming from* $\varphi_{p,n}$

$$\prod_{n \in T} (X_n)^{hC_n} \rightrightarrows \prod_p \prod_{n \in T/p} (X_{np}^{tC_p})^{hC_n}$$

In particular, $\mathrm{TR}(R, M) = \mathrm{TR}(\mathrm{THH}(R, M))$ *and* $\mathrm{TR}(R) = \mathrm{TR}(\mathrm{THH}(R))$.

Example 4.7.
(1) Suppose X is a polygonic spectra concetrated at degree n, then
$$\mathrm{TR}(X) = fib(X^{hC_n} \xrightarrow{can} X^{tC_n}) = X_{hC_n}$$
(2) If X is a cyclotomic spectra, viewed as a $T = \{1, p\}$ polygonic spectra with $X_1 = X_p = X$ and $\varphi_p : X_1 \to X_p^{tC_p}$ the Frobenius, then $\mathrm{TR}_T(X)$ is given by the pullback
$$\begin{array}{ccc} \mathrm{TR}_T(X) & \longrightarrow & X^{hC_p} \\ \big\downarrow & & \big\downarrow{\scriptstyle can} \\ X & \xrightarrow{\varphi_p} & X^{tC_p} \end{array}$$
Also denoted as X^{C_p}.

5. Adjunctions

Consider the restriction functor $\mathrm{res}_n : \mathrm{Sp}^{B\mathbb{T}} \to \mathrm{Sp}^{BC_n}$ which comes from the inclusion $\mathcal{C}_n \to \mathbb{T}$. Those combine to a functor $i : \mathrm{CycSp} \to \mathrm{PgcSp}$, where for a cyclotomic spectrum X we take $(iX)_n = \mathrm{res}_n X$, and $\varphi_{p,n} = \mathrm{res}_n \varphi_p$. For a general truncation set, we have $i_T : \mathrm{CycSp} \to \mathrm{PgcSp}_T$

the restriction map has both a left and right adjoint $\mathrm{ind}_n \dashv \mathrm{res}_n \dashv \mathrm{coind}_n$, given by

$$\mathrm{ind}_n(X) = X \otimes_{\Sigma^\infty C_n} \Sigma^\infty \mathbb{T}$$
$$\mathrm{coind}_n(X) = map_{Sp^{BC_n}}(\Sigma^\infty \mathbb{T}, X) = X^{hn\mathbb{Z}}$$

For a *finite* truncation set T, those induce left and right adjoints $L_T \dashv i_T \dashv R_T$ given by

$$L_T(X) = \bigoplus_{n \in T} \mathrm{ind}_n(X_n)$$
$$R_T(X) = \bigoplus_{n \in T} \mathrm{coind}_n(X_n)$$

with Frobenius maps coming from

$$\mathrm{ind}_n(X_n) \xrightarrow{\mathrm{ind}_n(\phi_{p,n})} \mathrm{ind}_n(X_{pn}^{tC_p}) \xrightarrow{\sim} \mathrm{ind}_n(X_{pn})^{tC_p}$$

The last equivalence comes from the fiber sequence $pn\mathbb{Z} \to \mathbb{Z} \to C_{pn} \to Bpn\mathbb{Z} = \mathbb{T}$, which implies $\mathrm{ind}_n = (-)_{h\mathbb{Z}}$, and $(-)_{h\mathbb{Z}}$ commutes with $(-)^{tC_p}$ as a finite limit. Similarly for coind.

Once we pass to $T = \mathbb{N}_{>0}$ (or generally an infinite truncation set), The formula for the left adjoint remains the same

$$L(X) = \bigoplus_{n>0} \mathrm{ind}_n(X_n)$$

But for the right adjoint we need to take a limit over finite truncation sets

$$R(X) = \lim_{T \subseteq_{fin} \mathbb{N}_{>0}} R_T(X|_T)$$

However, when X_n are uniformly bounded below, this simplifies to

$$R(X) = \prod_{n \in T} \mathrm{coind}_n(X_n)$$

Li, Ri : CycSp $\to$ CycSp can be described explicitly. Define $\widetilde{\mathrm{THH}}(\mathbb{S}[t]) = \mathrm{Fib}(\mathrm{THH}(\mathbb{S}[t]) \to \mathbb{S})$ where $\mathbb{S}[t] = \Sigma^\infty \mathbb{N}$.

Proposition 5.1.

$$Li(X) = X \otimes \widetilde{\mathrm{THH}}(\mathbb{S}[t])$$
$$Ri(X) = \Omega lim_n X \otimes \widetilde{\mathrm{THH}}(\mathbb{S}[t]/t_n)$$

Corollary 5.2. *Let X be a cyclotomic spectra, then*

$$\begin{aligned}\mathrm{TR}(X) := map_{\mathrm{PgcSp}}(\mathrm{THH}(\mathbb{S}), X) &= map_{\mathrm{CycSp}}(\widetilde{\mathrm{THH}}(\mathbb{S}[t]), X) \\ &= \Omega \lim_n \mathrm{TC}(X \otimes \widetilde{\mathrm{THH}}(\mathbb{S}[t]/t_n))\end{aligned}$$

This gives us a description of TR internal to cyclotomic spectra