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כ״ג בשבט ה׳תשפ״ב

On algebraically closed fields with a predicate

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תקציר

התזה נחלקת לשני פרקים, בשניהם נחקרת תורה של שדות סגורים אלגברית עם פרדיקט לתת-מבנה מסוים.

הפרק הראשון עוסק בתורה של זוגות (K, F) כאשר K שדה סגור אלגברית ו-F תת-שדה, ייתכן בעל מבנה נוסף. כחלק ממחקרם על שדות פסאודו-סגורים אלגברית, Cherlin, Van den Dries ו-Macintyre תיארו אינווריאנטים אלמנטריים לשדות אלו. אחד מאותם אינווריאנטים הוא התורה האלמנטרית של חבורת גלואה האבסולוטית, בשפה מתאימה בעלת ω סוגים. איווריאנט זה הוא כלי משמעותי לחקירת שדות פסאודו-סגורים אלגברית, והאינטואיציה שהמורכבות של שדה פסאודו-סגור אלגברית (מבחינת תורת המודלים) נשלטת על ידי התורה האלמנטרית של חבורת גלואה האבסולוטית שלו הוכיחה את עצמה במספר תוצאות מאז. למשל, Chatzidakis הוכיחה כי אם חבורת גלואה האבסולוטית היא NSOP_n $(n > 2)$ אז כך גם השדה. Ramsey הרחיב את התוצאה לתכונות NSOP_1 ו-NTP_1. ידוע כי חבורת גלואה האבסולוטית של שדה F ניתנת לפירוש בזוג (K, F) כאשר K שדה סגור אלגברית המרחיב את F. מתוך כך נולד אצלנו העניין בזוגות כאלו.

תוצאה מוקדמת על זוגות (K, F) כאשר K שדה סגור אלגברית ו-F תת-שדה ניתנה על ידי Keisler שהראה כי אם שדות F ו-F' שקולים אלמנטרית, אזי גם הזוגות (K, F) ו-(K, F') שקולים אלמנטרית. אם השדה F סגור אלגברית גם הוא, אז הזוג (K, F) הוא belle paire ובפרט יציב. בנוסף, כאשר F סגור אלגברית, Delon נתנה הרחבה של השפה עבורה לזוג (K, F) יש חילוץ כמתים.

לפרק זה קיימות שתי מטרות. האחת היא חקר תכונות תורת-המודליות בסיסיות של זוגות כאלו, כמו מודלים רווים, שלמות, חילוץ כמתים ושלמות-מודלית. השנייה היא הוכחת שימור של תכונות מתורת הסיווג מהשדה F לזוג (K, F). אנו נראה כי אם התורה של F היא (על-/ω-)יציבה/פשוטה/ NSOP_1/NIP אז כך גם התורה של (K, F). כמסקנה מיידית, נאמת שהתורה של שדה פסאודו-סגור אלגברית היא NSOP_1 אם ורק אם התורה האלמנטרית של חבורת גלואה האבסולוטית שלו היא NSOP_1. כמסקנות נוספות, אנו מסיקים כי כאשר F שדה פסאודו-סופי אז התורה של הזוג (K, F) היא פשוטה, ומסווגים את התורות של שרשראות של שדות סגורים אלגברית לפי טיפוס הסדר שלהן.

הפרק השני עוסק במחלקת המודלים הסגורית ישית של שדות עם פרדיקט לתת-מודול (מעל תת-חוג קבוע). למודולים סגורית ישית יש אופי שרירותי, או גנרי, מתוקף הגדרתם — כל מבנה סופי חסר כמתים שקיים באיזושהי הרחבה של המודל קיים גם במודל עצמו. מציאת אקסיומטיזציה מסדר ראשון למחלקת המודלים הסגורים ישית היא כלי חזק לחקירת המודלים הגנריים, ובמידה והתורה אינדוקטיבית אקסיומטיזציה זו תהיה העמית המודלי.

d'Elbée חקר את התורה של מודלים עם תת-מבנה גנרי, ובפרט הוא מצא עמית מודלי לשדות ממציין חיובי עם תת-מרחב וקטורי מעל תת-שדה סופי. בנוסף, הוא הגדיר יחסי אי-תלות חלשה ואי-תלות חזקה, ומצא תנאים למודל עם תת-מבנה גנרי להיות NSOP_1, כשבמקרה זה אי-תלות חלשה היא אי-תלות קים. שדות ממציין חיובי עם תת-מרחב וקטורי מעל תת-שדה סופי מקיימים את התנאים האלה, ולכן העמית המודלי שלהם הוא NSOP_1. בנוסף, נמצא כי העמית המודלי שלהם אינו פשוט.

טבעי לנסות להכליל תוצאה זו לשדות ממציין 0, או כאשר המרחב הוקטורי הוא מעל תת-שדה אינסופי. הכללה נוספת היא למודולים מעל תת-חוג אינסופי (חוג סופי הוא שדה). d'Elbée הוכיח כי למודלים הסגורים ישית של שדות ממציין 0 עם תת-חבורה אבלית (קרי, תת-מודול מעל $\mathbb{Z}$) אין אקסיומטיזציה מסדר ראשון, ובפרט לא קיים עמית מודלי. אף על פי כן, ניתן לחקור מודלים סגורים ישית של תורה אינדוקטיבית במסגרת לוגית שונה, הנקראת לוגיקת רובינסון, או קטגוריית המודלים הסגורים ישית. בתמצית, הכוונה היא שבמקום לחקור מודלים ושיכונים אלמנטריים ביניהם אנו חוקרים מודלים סגורים ישית ושיכונים בינהם. במסגרת זו חקרו Haykazyan ו- Kirby שדות עם אקספוננט — שדות F עם הומומורפיזם $E : F^+ \to F^\times$ מהחבורה החיבורית לחבורה הכפלית.

בפרק זה נלך בעקבות צעדיהם של Haykazyan ו- Kirby בעבור התורה של שדות עם תת-מודול. תחילה, ניתן תיאור של המודלים הסגורים ישית של שדות עם תת-מודול. תיאור זה לא יהיה מסדר ראשון באופן כללי, אך יהיה מסדר ראשון כאשר המציין חיובי ותת-המודול הוא מעל תת-חוג סופי. לאחר מכן, נוכיח כי קטגוריית המודלים הסגורים ישית של תורה זו היא NSOP_1, אך אינה NTP_2, ובפרט אינה פשוטה. ההוכחה של NSOP_1 תשתמש ביחס אי-התלות החלשה. בנוסף נחקור את יחס אי-התלות החזקה, ונראה שהוא מקיים n-תצרופת לכל n.

Abstract

The thesis is split into two chapters.

The first chapter is concerned with the model-theoretic study of pairs (K, F) where K is an algebraically closed field and F is a distinguished subfield of K allowing extra structure. We study the basic model-theoretic properties of those pairs, such as quantifier elimination, model-completeness and saturated models. We also prove some preservation results of classification-theoretic notions such as stability, simplicity, NSOP_1, and NIP. As an application, we conclude that a PAC field is NSOP_1 iff its absolute Galois group is (as a profinite group).

The second chapter deals with the class of existentially closed models of fields with a distinguished submodule (over a fixed subring). In the positive characteristic case, this class is elementary and was investigated by d'Elbée in [dE21a]. Here we study this class in Robinson's logic, meaning the category of existentially closed models with embeddings following Haykazyan and Kirby, and prove that in this context this class is NSOP_1 and TP_2.

Contents

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Chapter I

Algebraically closed fields with a distinguished subfield

I.1 Introduction

In their study of pseudo-algebraically closed fields, or PAC fields (known at that time as regularly closed fields, for obvious reasons, see Definition I.6.8) Cherlin, van den Dries and Macintyre [CvdDM80, CvdDM81] described elementary invariants for those fields. This was inspired by the work of Ax on pseudo-finite fields. Among those invariants is the elementary theory of the absolute Galois group of those fields in a suitable omega-sorted language, called the *inverse system of the absolute Galois group*. It was already clear to the authors of [CvdDM80, CvdDM81] that this invariant is an essential tool for the study of PAC fields. The intuition that the model theoretic complexity of the theory of PAC fields is mainly controlled by the theory of its absolute Galois group was confirmed by numerous results since then. For example, Chatzidakis [Cha19] proved that if the inverse system of the absolute Galois group of a PAC field is NSOP_n ($n > 2$), then so is the theory of the field. Ramsey [Ram18] proved the corresponding results for NTP_1 and NSOP_1. It is a fact that the inverse system of the absolute Galois group of a field F is interpretable in the theory of the pair (K, F) for any algebraically closed field K extending F (see [Cha02, Proposition 5.5]). This motivated our interest in the model-theoretic study of such pairs (K, F).

The model-theoretic study of pairs of fields goes back to Tarski when he raised in [Tar51] the question of the decidability of the pair $(\mathbb{R}, \mathbb{R} \cap \mathbb{Q}^{\mathrm{alg}})$ (the reals with a predicate for the reals algebraic over $\mathbb{Q}$). The (positive) answer was given by Robinson in [Rob59], who gave a full set of axioms for the theories of $(\mathbb{R}, \mathbb{R} \cap \mathbb{Q}^{\mathrm{alg}})$ and $(\mathbb{C}, \mathbb{Q}^{\mathrm{alg}})$. The celebrated work of Morley and of Shelah in the 70's created a growing interest in classification of first-order theories, and in particular of theories of fields and their expansions. It was known in the 80's that the theory of $(\mathbb{C}, \mathbb{Q}^{\mathrm{alg}})$ is stable[1] and Poizat [Poi83] generalized this result

[1] See the first sentence of [Poi83].

to a more general context: he gave a criterion for the stability of special pairs of elementary substructures $N \succ M$ (called "belle paires"), under a strong stability assumption on the theory of M (and N) called *nfcp*, introduced by Keisler [Kei67]. This was later generalized to the context of simple theories [BYPV03] with the notion of lovely pairs. Back to algebraically closed fields, Delon [Del12] introduced a language for quantifier elimination for pairs of algebraically closed fields and pairs of algebraically closed valued fields. Recently, Martin-Pizarro and Ziegler [MPZ20] proved that the theory of proper pairs of algebraically closed fields is equational, by a deep analysis of definable sets.

As was mentioned above, the main topic of this chapter is another generalization of pairs of algebraically closed fields, which are pairs (K, F) where F is an arbitrary field, perhaps with some extra structure (in a language extending the language of fields), and $K \supset F$ is an algebraically closed infinite extension. An early result about this theory was given by Keisler [Kei64]: if F and F' are two elementarily equivalent fields (not real-closed nor algebraically closed and without extra structure), then the pairs (K, F) and (K', F') are also elementarily equivalent, for any algebraically closed extensions $K \supset F$, $K' \supset F'$. In [HKR18], Hils, Kamensky and Rideau gave a quantifier elimination result for the theory of the pairs (K, F), which we also obtain in Theorem I.4.3. We were not aware of this result while writing the proof and we decided to keep our proof for completeness.

The purpose of this chapter is twofold. For one, we are interested in the basic properties of the theory of the pairs such as saturated models, completeness, quantifier elimination and model-completeness. For example, as we mentioned above we prove quantifier elimination for the theory of pairs (K, F) (see Theorem I.4.3) in a natural expansion of the language following Delon's approach [Del12]. This allows us to isolate a condition implying the model-completeness of the theory of the pair (K, F) which is weaker than the model completeness of the theory of F (see Theorem I.4.12). Secondly, we prove preservation of several classification-theoretic properties: if the theory of F is (ω-/super) stable/NIP/simple/NSOP$_1$, then so is the theory of the pair (K, F) (see Corollaries I.5.25 and I.5.26 and Theorems I.5.9, I.5.13, I.5.16 and I.5.34). In the case of NSOP$_1$, we also identify Kim-independence for algebraically closed sets (see Proposition I.5.11).

As immediate applications we conclude that the theory of a PAC field F in the language of rings is NSOP$_1$ iff the theory of its Galois group is (see Proposition I.6.7) and prove that when F is pseudofinite in the language of rings the theory of the pair (K, F) is simple. In addition, we consider the theory ACF^I of a chain of algebraically closed fields ordered by some linear order I, and discuss its properties depending on the order type of I (see Proposition I.6.4).

I.2 Preliminaries

In this section we present common definitions and results from fields and model theory. We will start by setting up some basic notation for the whole paper.

Notation I.2.1. *Whenever A is a field, let $\overline{A}$ be its algebraic closure. Whenever A and B are subfields of a larger field, let $A.B$ be their field compositum. If A is a field and S is a set, then let $A(S)$ be the field extension of A by the elements of S. Say that the set S is algebraically independent over A if each element $s \in S$*

is algebraically independent over $A(S \setminus \{s\})$. If R is a sub-ring of a larger field, then denote by $\mathrm{Frac}(R)$ *the field generated by R. Unless specified otherwise, all the fields will be subfields of a large algebraically closed field.*

I.2.1 Linear disjointness

Definition I.2.2. Let A, B and C be fields with $C \subseteq A \cap B$.

1. Say that A is *linearly disjoint* from B over C if whenever $a_0, \dots, a_{n-1} \in A$ are linearly independent over C they are also linearly independent over B. Denote this by $A \mathop{\smile\!\!\!\!\!\!|}^{l}_{C} B$.

2. Say that A is *algebraically disjoint* from B over C if whenever $a_0, \dots, a_{n-1} \in A$ are algebraically independent over C, then they are also algebraically independent over B. This is the same as the non-forking independence in ACF, which we will denote $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{ACF}}_{C} B$.

Fact I.2.3 ([Mor96, Proposition 20.2])**.** *Let A, B and C be fields with $C \subseteq A \cap B$. Construct a map $A \otimes_C B \to A[B]$ by mapping $a \otimes b \mapsto ab$. This map is an isomorphism iff $A \mathop{\smile\!\!\!\!\!\!|}^{l}_{C} B$.*

Fact I.2.4. *The following is a list of useful model theoretic properties that $\mathop{\smile\!\!\!\!\!\!|}^{l}$ has inside* ACF. *Let A, B, C, D, A', B' and C' be fields with $C \subseteq A \cap B$, $C' \subseteq A' \cap B'$ and $B \subseteq D$.*

- *(Invariance) if $ABC \equiv A'B'C'$ and $A \mathop{\smile\!\!\!\!\!\!|}^{l}_{C} B$, then $A' \mathop{\smile\!\!\!\!\!\!|}^{l}_{C'} B'$.*
- *(Monotonicity) if $A \mathop{\smile\!\!\!\!\!\!|}^{l}_{C} D$, then $A \mathop{\smile\!\!\!\!\!\!|}^{l}_{C} B$.*
- *(Base monotonicity) if $A \mathop{\smile\!\!\!\!\!\!|}^{l}_{C} D$, then $A.B \mathop{\smile\!\!\!\!\!\!|}^{l}_{B} D$.*
- *(Transitivity) if $A \mathop{\smile\!\!\!\!\!\!|}^{l}_{C} B$ and $A.B \mathop{\smile\!\!\!\!\!\!|}^{l}_{B} D$, then $A \mathop{\smile\!\!\!\!\!\!|}^{l}_{C} D$.*
- *(Symmetry) if $A \mathop{\smile\!\!\!\!\!\!|}^{l}_{C} B$, then $B \mathop{\smile\!\!\!\!\!\!|}^{l}_{C} A$.*
- *(Stationarity) if $A \equiv_C A'$ and $A \mathop{\smile\!\!\!\!\!\!|}^{l}_{C} B$, $A' \mathop{\smile\!\!\!\!\!\!|}^{l}_{C} B$, then $A \equiv_B A'$.*
- *(Local character) for a finite tuple a, there exists a countable subfield $B_0 \subseteq B$, such that $B_0(a) \mathop{\smile\!\!\!\!\!\!|}^{l}_{B_0} B$.*

Proof. Invariance is trivial. Proofs for monotonicity, base monotonicity and transitivity can be found in [FJ08, Lemma 2.5.3], symmetry is proven in [FJ08, Lemma 2.5.1]. Stationarity follows directly from Fact I.2.3 and quantifier elimination in ACF.

Local character follows from [Lan72, Theorem III.7, Proposition III.6 and Theorem III.8], by setting B_0 to be the field of definition of the locus of a over B. This gives an even stronger result, as B_0 is finitely generated and not merely countable. For a more direct proof of local character, see Remark I.5.2. ☐

Corollary I.2.5. *Let A_0, B_0, C_0, A_1, B_1 and C_1 be fields with $C_0 \subseteq A_0 \cap B_0$, $C_1 \subseteq A_1 \cap B_1$, such that $A_0 \mathop{\smile\!\!\!\!\!\!|}^{l}_{C_0} B_0$, $A_1 \mathop{\smile\!\!\!\!\!\!|}^{l}_{C_1} B_1$. Suppose there are isomorphism $f : A_0 \to A_1$, $g : B_0 \to B_1$ such that $f|_{C_0} = g|_{C_0}$. Then there is a unique isomorphism $F : A_0.B_0 \to A_1.B_1$ such that $F|_{A_0} = f$, $F|_{B_0} = g$.*

Proof. Consider A_0, A_1, B_0 and B_1 as tuples, such that f and g match the tuples. Extend g to an automorphism σ arbitrarily. From invariance, by applying σ to $A_0 \mathop{\smile\!\!\!\!\!\!\!\;|}^l_{C_0} B_0$, we get $\sigma(A_0) \mathop{\smile\!\!\!\!\!\!\!\;|}^l_{C_1} B_1$. From stationarity $\sigma(A_0) \equiv_{B_1} A_1$, let τ be an automorphism witnessing the equivalence. Let $F = (\tau \circ \sigma)|_{A_0.B_0}$, we have $F(A_0) = \tau(\sigma(A_0)) = A_1$ and $F(B_0) = \tau(\sigma(B_0)) = \tau(B_1) = B_1$ as tuples. In particular, $F : A_0.B_0 \to A_1.B_1$ is an isomorphism, and from the way we chose the tuples $F|_{A_0} = f$ and $F|_{B_0} = g$. □

Definition I.2.6. A field extension $A \subseteq B$ is called:

- *regular* if $\overline{A} \mathop{\smile\!\!\!\!\!\!\!\;|}^l_A B$,
- *separable* if $A^{1/p} \mathop{\smile\!\!\!\!\!\!\!\;|}^l_A B$, where $p = \mathrm{char}(A) > 0$ and $A^{1/p}$ is the field of p-th roots of all elements in A (if $\mathrm{char}(A) = 0$, then all extensions are separable), and
- *relatively algebraically closed* if $\overline{A} \cap B = A$.

Fact I.2.7. *Suppose $A \subseteq B$ is a field extension.*

1. *[FJ08, Lemma 2.6.4] The extension $A \subseteq B$ is regular iff it is separable and relatively algebraically closed.*
2. *[FJ08, Lemma 2.6.7] If the extension $A \subseteq B$ is regular and C is a field extending A such that $B \mathop{\smile\!\!\!\!\!\!\!\;|}^{\mathrm{ACF}}_A C$, then $B \mathop{\smile\!\!\!\!\!\!\!\;|}^l_A C$.*

Lemma I.2.8. *If $A \subseteq B$ is a regular field extension and $\sigma : B \to B'$ is an isomorphism of fields, then $\sigma(A) \subseteq B'$ is regular.*

Proof. We can extend σ to the algebraic closure, $\tilde{\sigma} : \overline{B} \to \overline{B'}$. From $\overline{A} \mathop{\smile\!\!\!\!\!\!\!\;|}^l_A B$ we get by invariance $\tilde{\sigma}(\overline{A}) \mathop{\smile\!\!\!\!\!\!\!\;|}^l_{\sigma(A)} B'$. But $\tilde{\sigma}(\overline{A}) = \overline{\sigma(A)}$, so we have $\overline{\sigma(A)} \mathop{\smile\!\!\!\!\!\!\!\;|}^l_{\sigma(A)} B'$ as needed. □

Lemma I.2.9. *If $A \subseteq B$ is a regular field extension and S is a set algebraically independent over B, then $\overline{A(S)} \mathop{\smile\!\!\!\!\!\!\!\;|}^l_A B$.*

Proof. As S is algebraically independent over B, we have $\overline{A(S)} \mathop{\smile\!\!\!\!\!\!\!\;|}^{\mathrm{ACF}}_A B$. Fact I.2.7(2) implies that $\overline{A(S)} \mathop{\smile\!\!\!\!\!\!\!\;|}^l_A B$. □

I.2.2 Language of regular extensions

In [Mac08], Macintyre defines relations in the language of rings that are preserved in a field extension iff it is regular. We will present those relations, and use them to expand a theory of fields[2] in such a way that the models are the same but for any two models M, N, N extends M iff it is a regular field extension.

Fact I.2.10 ([Mac08, §4.7])**.** *Let $A \subseteq B$ be a field extension.*

1. *The extension is relatively algebraically closed iff it preserves the relations* $\mathrm{Sol}_n(x_0, \ldots, x_{n-1}) = \exists y(x_0 + x_1 y + \cdots + x_{n-1} y^{n-1} + y^n = 0)$ *for $n \geq 1$.*

[2]By a *theory of fields*, we mean a theory in a language expanding the language of rings which contains all the fields axioms.

2. *For $p = \mathrm{char}(A)$, the extension is separable iff it preserves the relations $D_{n,p}(x_0, \dots, x_{n-1}) = \exists y_0, \dots, y_{n-1}(y_0^p x_0 + \cdots + y_{n-1}^p x_{n-1} = 0)$ for $n \geq 1$ (note that if $p = 0$, $D_{n,p}$ is quantifier-free definable).*

Corollary I.2.11. *Suppose M and N are fields. If $M \prec N$, then $M \subseteq N$ is a regular extension.*

Proof. The fact that $M \prec N$ implies in particular that $M \subseteq N$ is a field extension that preserves Sol_n and $D_{n,p}$ ($p = \mathrm{char}(A)$). By Fact I.2.10 the extension $M \subseteq N$ is relatively algebraically closed and separable, so by Fact I.2.7(1) it is a regular extension. □

Definition I.2.12. Let T be a theory of fields in a language L expanding the language of rings. Define $L_{\mathrm{reg}} = L \cup \{\mathrm{Sol}_n\}_{n\geq 1} \cup \left\{\tilde{D}_{n,p}\right\}_{n\geq 1, p \in \mathrm{Primes}\cup\{0\}}$, where Sol_n, $\tilde{D}_{n,p}$ are n-ary relations, and extend T to T_{reg} in L_{reg} by defining $\mathrm{Sol_n}$ as above and defining

$$\tilde{D}_{n,p} = D_{n,p} \wedge (\underbrace{1 + \cdots + 1}_{p} = 0).$$

Lemma I.2.13. *Let T be a theory of fields and let $Q, R \models T$ with $Q \subseteq R$ a substructure. By adding definable relations, Q and R can be expanded to models of T_{reg}. Then Q is an L_{reg}-substructure of R iff $Q \subseteq R$ is a regular field extension.*

Proof. Let $p = \mathrm{char}(Q)$. Note that by Facts I.2.7 and I.2.10, it is enough to prove that Q is an L_{reg}-substructure of R iff the extension $Q \subseteq R$ preserves Sol_n and $D_{n,p}$ for all n. Indeed, this equivalence holds because $\tilde{D}_{n,p}$ is equivalent to $D_{n,p}$ and $\tilde{D}_{n,q}$ is trivially false for any prime $q \neq p$. □

I.2.3 NSOP$_1$

In this subsection we will review the definition and basic properties of NSOP$_1$ theories.

We will work in a monster model $\mathbb{M}$ (large, saturated) of a complete theory T.

Definition I.2.14. A formula $\phi(x; y)$ has SOP$_1$ if there is a tree of tuples $(b_\eta)_{\eta \in 2^{<\omega}}$ such that

- for all $\eta \in 2^\omega$, $\left\{\phi(x; b_{\eta|\alpha}) \mid \alpha < \omega\right\}$ is consistent,
- for all $\eta \in 2^{<\omega}$, if $\nu \trianglerighteq \eta \frown \langle 0 \rangle$, then $\left\{\phi(x; b_\nu), \phi(b; a_{\eta \frown \langle 1 \rangle})\right\}$ is inconsistent.

We say that a theory T is SOP$_1$ if some formula has SOP$_1$ modulo T. Otherwise, T is NSOP$_1$.

Definition I.2.15. Let A be a set and a and b tuples, say that a is *coheir independent* of b over A if the type $\mathrm{tp}(a/Ab)$ is finitely satisfiable in A, and denote $a \mathop{\smile\!\!\!\!\!\!\mid}^u_A b$. A sequence $(a_i)_{i\in I}$ is an A-indiscernible coheir sequence if it is A-indiscernible and $a_i \mathop{\smile\!\!\!\!\!\!\mid}^u_A a_{<i}$

Using coheir-independence, we can use a different criterion for NSOP$_1$, due to [CR16, Theorem 5.7].

Fact I.2.16 (Weak independent amalgamation). *The theory T is NSOP$_1$ iff given any model $M \models T$ and tuples $a_0b_0 \equiv_M a_1b_1$ such that $b_1 \mathop{\smile\!\!\!\!\!\mid}^u_M b_0$ and $b_i \mathop{\smile\!\!\!\!\!\mid}^u_M a_i$ for $i = 0, 1$, there exists a such that $ab_0 \equiv_M ab_1 \equiv_M a_0b_0$.*

Kim-dividing, and its extension Kim-forking, were defined in [KR20], over arbitrary sets. For our purposes we will give a simplified definition, which we will call Kim^u-dividing, and define it only over models.

Definition I.2.17. A formula $\phi(x, b)$ *Kim^u-divides* over a model M if there exists an M-indiscernible coheir sequence $(b_i)_{i<\omega}$ with $b \equiv_M b_i$, such that $\{\phi(x, b_i)\}_{i<\omega}$ is inconsistent. A formula *Kim^u-forks* over M if it implies a disjunction of Kim^u-dividing formulas over M.

A type Kim^u-divides (Kim^u-forks) over M if it implies a Kim^u-dividing (Kim^u-forking) formula over M. Denote $a \mathop{\smile\!\!\!\!\!\mid}^K_M b$ when the type $\mathrm{tp}(a/Mb)$ does not Kim^u-fork over M.

Remark I.2.18. *In this definition, $(b_i)_{i<\omega}$ is a Morley sequence in a restriction of a global coheir type. In the original definition of Kim-dividing, the global coheir type is replaced with a global invariant type. By Kim's lemma for Kim-dividing [KR20, Theorem 3.16], those definitions are equivalent for NSOP$_1$ theories.*

Remark I.2.19. *The type $\mathrm{tp}(a/Mb)$ does not Kim^u-divide over M iff for every M-indiscernible coheir sequence $(b_i)_{i<\omega}$ with $b \equiv_M b_i$, there exists a' such that $ab \equiv_M a'b_i$ for every $i < \omega$.*

Fact I.2.20. *Suppose T is NSOP$_1$, then*

1. *[KR20, Theorem 3.16] If $\phi(x, b)$ Kim-divides over $M \models T$, then for* every *M-indiscernible coheir sequence $(b_i)_{i<\omega}$ with $b \equiv_M b_i$, $\{\phi(x, b_i)\}_{i<\omega}$ is inconsistent.*
2. *[KR20, Proposition 3.19] Kim-dividing is equivalent to Kim-forking over models.*
3. *[KR20, Theorem 5.16] $\mathop{\smile\!\!\!\!\!\mid}^K$ is symmetric over models.*
4. *[KR20, Corolary 5.17] Let $M \models T$, $a \mathop{\smile\!\!\!\!\!\mid}^K_M b \iff \mathrm{acl}(a) \mathop{\smile\!\!\!\!\!\mid}^K_M b \iff a \mathop{\smile\!\!\!\!\!\mid}^K_M \mathrm{acl}(b)$.*
5. *[KR20, Proposition 8.8] T is simple iff $\mathop{\smile\!\!\!\!\!\mid}^K$ satisfies base monotonicity over models: if $M, N \models T$ and $M \subseteq N$, then $a \mathop{\smile\!\!\!\!\!\mid}^K_M Nb$ implies $a \mathop{\smile\!\!\!\!\!\mid}^K_N b$.*
6. *[KR20, Proposition 8.4] T is simple iff $\mathop{\smile\!\!\!\!\!\mid}^K = \mathop{\smile\!\!\!\!\!\mid}^f$ over models.*

I.3 Basic properties of ACF$_T$

In this section we will define and study the basic properties of ACF$_T$, the theory of algebraically closed fields with a distinguished subfield (in an arbitrary language). We will also consider expansions of the theory by definable relations and functions, that Delon defined to study pairs of ACF in [Del12].

I.3.1 Delon's language

Definition I.3.1. Let T be a theory of fields (not necessarily complete), in a language expanding the language of rings $L \supseteq L_{\text{rings}}$. Expand L to the language $L^P = L \cup \{P\}$, with P a unitary predicate, and expand ACF to ACF_T in the language L^P by adding the following axioms:

1. P is a model of T. This can be achieved by taking all the axioms of T and restricting the quantifiers to be over P.
2. For every n-ary function symbol $f \in L \setminus L_{\text{rings}}$, if $x_0, \dots x_{n-1} \in P$, then $f(x_0, \dots, x_{n-1}) \in P$. Else, if some $x_i \notin P$, then we do not care about the value of $f(x_0, \dots, x_{n-1})$, and we can set it arbitrarily to 0.
3. For every n-ary relation symbol $R \in L \setminus L_{\text{rings}}$, if some $x_i \notin P$, then $\neg R(x_0, \dots, x_{n-1})$. That is, $R \subseteq P^n$.
4. The degree of the field extension of the whole model over P is infinite, i.e. the model has infinite dimension as a vector space over P. By the Artin-Schreier theorem [AS27], it is enough to assert that the degree is at least 3.

Remark I.3.2. *The assumption that the degree of the model over P is infinite, that is, for $M \models \text{ACF}_T$, $[M : P_M] = \infty$, always holds when models of T are not algebraically closed or real closed, because in that case $[\overline{P_M} : P_M] = \infty$. When models of T are algebraically closed, it simply means that $M \neq P_M$, i.e. (M, P_M) is a proper pair. The only case excluded is when models of T are real closed and $M = \overline{P_M}$, but then $(\overline{P_M}, P_M)$ is definable in P_M.*

Definition I.3.3. Let T, L be as above. Consider the following definable relations and functions over ACF_T:

- For $n \geq 1$, define the n-ary relation l_n by $l_n(x_0, \dots, x_{n-1})$ iff $x_0, \dots, x_{n-1}$ are linearly independent over P.
- For $n \geq 1$, suppose we have $l_n(x_0, \dots, x_{n-1})$ and $\neg l_{n+1}(x_0, \dots, x_n)$. That is, $x_0, \dots, x_{n-1}$ are linearly independent over P and x_n is in their span over P. Then, there are unique $y_i \in P$ such that $x_n = y_0 x_0 + \cdots + y_{n-1} x_{n-1}$. Define the $n+1$-ary function $f_{n,i}$ by $f_{n,i}(x_n; x_0, \dots, x_{n-1}) = y_i$. If $x_0, \dots, x_n$ do not satisfy this condition, then we do not care about the value of $f_{n,i}(x_n; x_0, \dots, x_{n-1})$ and can set it arbitrarily to 0.

Expand ACF_T to ACF_T^{ld} in the language $L^{ld} = L^P \cup \{l_n\}_{n \geq 1}$, by defining l_n as above. Expand ACF_T^{ld} to ACF_T^f in the language $L^f = L^{ld} \cup \{f_{n,i}\}_{n > i \geq 0}$, by defining $f_{n,i}$ as above.

Notation I.3.4. *If $M \models \text{ACF}_T$, then let P_M be the predicate P in M with the associated L-structure. If $A \subseteq M$ is a subset, then let $P_A = P_M \cap A$. This notation is used instead of the usual $P(M)$ and $P(A)$, because the notation $P(A)$ is reserved for the field extension of P by A.*

Definition I.3.5. Call a formula $\phi(x) \in L^P$ *bounded* if every quantifier in ϕ is over P.

Remark I.3.6. *For a formula $\phi(x) \in L$ there is a corresponding bounded formula $\phi^P(x) \in L^P$ created by restricting every quantifier to be over P and asserting $x \in P$. For $M \models \text{ACF}_T$, we have $\phi^P(M) = \phi(P_M)$.*

I.3.2 Substructures and isomorphisms

Lemma I.3.7. *Let $M \models \mathrm{ACF}_T^f$ and $A \subseteq M$ a subset. Then A is an L^f-substructure iff $P_A \subseteq P_M$ is an L-substructure, A is a subring, P_A is a subfield and* $\mathrm{Frac}(A) \mathop{\smile\!\!\!\!\!\!\mid}^l_{P_A} P_M$.

Proof. Suppose $A \subseteq M$ is an L^f-substructure. We get that $P_A \subseteq P_M$ is an L-substructure, because for any function symbol $f \in L$ and $\overline{a} \in P_A$, $f(\overline{a}) \in A$ as $A \subseteq M$ is a substructure, and also $f(\overline{a}) \in P_M$ because of the axioms of ACF_T, so $f(\overline{a}) \in A \cap P_M = P_A$. It is clear that A is a subring, and so is P_A, but for every $0 \neq a \in P_A$, $a^{-1} = f_{1,0}(1; a) \in P_A$, so P_A is also a subfield. By [Lan72, Chapter III, Criterion 1], to prove that $\mathrm{Frac}(A) \mathop{\smile\!\!\!\!\!\!\mid}^l_{P_A} P_M$, it is enough to show that if $a_0, \dots, a_{n-1} \in A$ are linearly dependent over P_M, then they are linearly dependent over P_A. Suppose $a_0, \dots, a_{n-1} \in A$ are linearly dependent over P_M. If $a_0 = 0$, then the tuple is trivially linearly dependent over P_A. Else, there is some maximal $1 \leq k < n$ such that $a_0, \dots, a_{k-1}$ are linearly independent over P_M, so we have $\models l_k(a_0, .., a_{k-1})$ and $\models \neg l_{k+1}(a_0, \dots, a_k)$. Hence we can look at $p_i = f_{k,i}(a_k; a_0, \dots, a_{k-1}) \in P_M$, which give us $a_k = p_0 a_0 + \cdots + p_{k-1} a_{k-1}$. Because A is a substructure, $p_i \in A$, so $p_i \in P_A$. Thus, $a_0, \dots, a_{n-1}$ are linearly dependent over P_A.

In the other direction, suppose A is a subring, P_A is a subfield, $P_A \subseteq P_M$ is an L-substructure and $\mathrm{Frac}(A) \mathop{\smile\!\!\!\!\!\!\mid}^l_{P_A} P_M$. It follows that $\mathrm{Frac}(A) \cap P_M = P_A$, and in particular $A \cap P_M = P_A$. For any function symbol $f \in L \setminus L_{\mathrm{rings}}$ and $a_0, \dots, a_{n-1} \in A$, if $a_0, \dots, a_{n-1} \in P_A$, then $f(a_0, \dots, a_{n-1}) \in P_A$ as $P_A \subseteq P_M$ is a substructure, and else we defined $f(a_0, \dots, a_{n-1}) = 0 \in A$. It remains to check that A is closed under $f_{n,i}$. Let $a_0, \dots, a_n \in A$ and suppose $\models l_n(a_0, \dots, a_{n-1})$, $\models \neg l_{n+1}(a_0, \dots, a_n)$. Let $p_i = f_{n,i}(a_n; a_0, \dots, a_{n-1})$, that is $p_i \in P_M$ and $a_n = p_0 a_0 + \cdots + p_{n-1} a_{n-1}$. We know that $a_0, \dots, a_n$ are linearly dependent over P_M, so by $\mathrm{Frac}(A) \mathop{\smile\!\!\!\!\!\!\mid}^l_{P_A} P_M$ they are linearly dependent over P_A. However, $a_0, \dots, a_{n-1}$ must be linearly independent over P_A, as they are linearly independent over P_M, so a_n can be written as a linear combination of $a_0, \dots, a_{n-1}$ over P_A. This linear combination is in particular over P_M, but $a_n = p_0 a_0 + \cdots + p_{n-1} a_{n-1}$ is the unique linear combination over P_M, so we must have $p_0, \dots, p_{n-1} \in P_A$, as needed. □

Corollary I.3.8. *If $M \models \mathrm{ACF}_T^f$ and $A \subseteq M$ is an L^f-substructure, then* $\mathrm{Frac}(A) \subseteq M$ *is an L^f-substructure with* $P_{\mathrm{Frac}(A)} = P_A$.

Proof. Lemma I.3.7 implies that $\mathrm{Frac}(A) \mathop{\smile\!\!\!\!\!\!\mid}^l_{P_A} P_M$, and in particular $P_{\mathrm{Frac}(A)} = P_M \cap \mathrm{Frac}(A) = P_A$. Thus, $P_{\mathrm{Frac}(A)} \subseteq P_M$ is a subfield and an L-substructure, $\mathrm{Frac}(A)$ is a subring (even subfield) and $\mathrm{Frac}(A) \mathop{\smile\!\!\!\!\!\!\mid}^l_{P_{\mathrm{Frac}(A)}} P_M$, so by Lemma I.3.7 $\mathrm{Frac}(A) \subseteq M$ is an L^f-substructure. □

Lemma I.3.9. *Let $M, N \models \mathrm{ACF}_T^f$ and let $A \subseteq M$, $B \subseteq N$ be L^f-substructures. A map $\sigma : A \to B$ is an L^f-isomorphism iff σ is an isomorphism of rings such that $\sigma(P_A) = P_B$ and $\sigma|_{P_A} : P_A \to P_B$ is an L-isomorphism.*

Proof. If σ is an L^f isomorphism, then it is clearly an isomorphism of rings, $\sigma(P_A) = P_B$ because σ preserves P and $\sigma|_{P_A} : P_A \to P_B$ is an L-isomorphism because L^f expands L on P. For the other direction, we need to show that σ preserves l_n, $f_{n,i}$. Let $a_0, \dots, a_{n-1} \in A$ with $\models l_n(a_0, \dots, a_{n-1})$. Suppose we have $\models \neg l_n(\sigma(a_0), \dots, \sigma(a_{n-1}))$, i.e. $\sigma(a_0), \dots, \sigma(a_{n-1})$ are linearly dependent over P_N. Lemma I.3.7 implies that $\mathrm{Frac}(B) \mathop{\smile\!\!\!\!\!\!|}^{l}_{P_B} P_N$, so $\sigma(a_0), \dots, \sigma(a_{n-1})$ are also linearly dependent over P_B. There are $q_0, \dots, q_{n-1} \in P_B$ such that $q_0\sigma(a_0) + \dots + q_{n-1}\sigma(a_{n-1}) = 0$ By applying σ^{-1} we get $\sigma^{-1}(q_0)a_0 + \dots + \sigma^{-1}(q_{n-1})a_{n-1} = 0$, however $\sigma^{-1}(q_0), \dots, \sigma^{-1}(q_{n-1}) \in P_A$, in contradiction to $\models l_n(a_0, \dots, a_{n-1})$. The other direction follows from symmetry. Now suppose we have $a_0, \dots, a_n \in A$ with $\models l_n(a_0, \dots, a_{n-1})$ and $\models \neg l_{n+1}(a_0, \dots, a_n)$. By the first part, we also have $\models l_n(\sigma(a_0), \dots, \sigma(a_{n-1}))$ and $\models \neg l_{n+1}(\sigma(a_0), \dots, \sigma(a_n))$. Let $p_i = f_{n,i}(a_n; a_0, \dots, a_{n-1}) \in P_A$, $a_n = p_0 a_0 + \dots + p_{n-1}a_{n-1}$. Apply σ to get $\sigma(a_n) = \sigma(p_0)\sigma(a_0) + \dots + \sigma(p_{n-1})\sigma(a_{n-1})$, but $\sigma(p_0), \dots, \sigma(p_{n-1}) \in P_B$, so by uniqueness $\sigma(p_i) = f_{n,i}(\sigma(a_n); \sigma(a_0), \dots, \sigma(a_{n-1}))$. ☐

Lemma I.3.10. *Let $M, N \models \mathrm{ACF}_T$. By adding definable relations and functions, M and N can be expanded to models of ACF_T^{ld}, ACF_T^{f}. With those expansions, the following are equivalent:*

1. *$M \subseteq N$ is an L^f-substructure.*
2. *$M \subseteq N$ is an L^{ld}-substructure.*
3. *$M \subseteq N$ is a subfield, $P_M \subseteq P_N$ is an L-substructure and $M \mathop{\smile\!\!\!\!\!\!|}^{l}_{P_M} P_N$.*

Proof. $1 \implies 2$: L^{ld} is a restriction of L^f.

$2 \implies 3$: It is clear that $M \subseteq N$ is a subfield and $P_M \subseteq P_N$ as sets. For every quantifier free formula $\phi(\overline{x}) \in L$ and $\overline{a} \in P_M$, $P_M \models \phi(\overline{a}) \iff M \models \phi(\overline{a}) \wedge \overline{a} \in P \iff N \models \phi(\overline{a}) \wedge \overline{a} \in P \iff P_N \models \phi(\overline{a})$, so P_M is an L-substructure of P_N. Let $a_0, \dots, a_{n-1} \in M$ be linearly independent over P_M, $M \models l_n(a_0, \dots, a_{n-1}) \implies N \models l_n(a_0, \dots, a_{n-1})$, so $a_0, \dots, a_{n-1}$ are linearly independent over P_N. Thus, $M \mathop{\smile\!\!\!\!\!\!|}^{l}_{P_M} P_N$.

$3 \implies 1$: Let M' be the L^f-structure with the same underlying set as M, but with structure induced as a subset of N. Note that $M' \subseteq N$ is really an L^f-substructure, from Lemma I.3.7. To prove that M is an L^f-substructure of N, we need to show that M and M' have the same structure, that is that the identity map $id : M \to M'$ is an L^f-isomorphism. We know that M is a subfield of N, so $id : M \to M'$ is a field isomorphism. From $M \mathop{\smile\!\!\!\!\!\!|}^{l}_{P_M} P_N$ we get that $P_{M'} = M \cap P_N = P_M$ and P_M is an L-substructure of P_N, so $id|_{P_M} : P_M \to P_{M'}$ is an L-isomorphism. Lemma I.3.9 implies that id is an L^f-isomorphism. ☐

I.3.3 Saturated models

We will study saturated models of ACF_T. Note that κ-saturated models of ACF_T are the same as κ-saturated models of ACF_T^{ld} or ACF_T^{f}, because $\{l_n\}_{n>1}$ and $\{f_{n,i}\}_{n>i>0}$ are definable in ACF_T. A full characterization of κ-saturated models will be given in Proposition I.4.11.

Lemma I.3.11. *If $M \models \mathrm{ACF}_T$ is κ-saturated, then P_M is a κ-saturated model of T.*

Proof. Follows from Remark I.3.6, by relativizing each formula in the type we wish to realize to P. ☐

For the next result, we will need the following algebraic technical lemma, whose proof is left as an exercise to the reader.

Fact I.3.12. *Suppose F is a field and t is transcendental over F. For every n, $[F(t):F(t^n)] = n$.*

Lemma I.3.13. *If $M \models \mathrm{ACF}_T$ is κ-saturated, then* $\mathrm{trdeg}(M/P_M) \geq \kappa$.

Proof. Let $S \subseteq M$ be an algebraically independent set over P_M. Suppose $|S| < \kappa$, we want to prove that there is some $a \in M$ such that $a \notin \overline{P_M(S)}$. Consider the partial type over S

$$\Sigma(x) = \{\forall \bar{y} \in P \ (q(x,\bar{y}) = 0 \to \forall x' q(x',\bar{y}) = 0) \mid q(x,\bar{y}) \in Q[x,\bar{y},S]\}$$

where Q is the prime field ($\mathbb{F}_p$ or $\mathbb{Q}$), x is a single variable and $\bar{y}$ is a tuple of variables. Let $\Sigma_n(x)$ contain all formulas in $\Sigma(x)$ where the degree of $q(x,\bar{y})$ in x is $\leq n$. We will show that $a \models \Sigma_n(x)$ iff $[P_M(S,a) : P_M(S)] > n$ and that $\Sigma_n(x)$ is satisfiable in M. From compactness and saturation ($|S| < \kappa$), we will get that $\Sigma(x)$ is satisfied by some $a \in M$. But then $[P_M(S,a) : P_M(S)] > n$ for all n, so $a \notin \overline{P_M(S)}$.

Suppose $a \models \Sigma_n(x)$. If $[P_M(S,a) : P_M(S)] \leq n$, then there is some non-zero polynomial $r(x) \in P_M(S)[x]$ of degree $\leq n$ such that $r(a) = 0$. The coefficients of $r(x)$ are rational functions in S over P_M. By multiplying by the denominators, we can assume the coefficients are polynomials in S and P_M, so $r(x) = q(x,\bar{p})$ for $q(x,\bar{y}) \in Q[x,\bar{y},S]$ and $\bar{p} \in P_M$. However, because $q(a,\bar{p}) = r(a) = 0$, we get from $a \models \Sigma_n(x)$ that $r(x)$ is constant zero.

Now suppose $[P_M(S,a) : P_M(S)] > n$. Let $q(x,\bar{y}) \in Q[x,\bar{y},S]$ of degree $\leq n$ in x and $\bar{p} \in P_M$, such that $q(a,\bar{p}) = 0$. The polynomial $q(x,\bar{p})$ is over $P_M(S)$, has degree $\leq n$ and has a as root, but $[P_M(S,a) : P_M(S)] > n$, so $q(x,\bar{p})$ must be constant zero. Hence $a \models \Sigma_n(x)$.

To prove that $\Sigma_n(x)$ is satisfiable for every n, we need to prove that there is some $a \in M$ such that $[P_M(S,a) : P_M(S)] > n$. Split into three cases.

1. $S = \emptyset$, $M \neq \overline{P_M}$: Take some $a \in M \setminus \overline{P_M}$ and we are done.

2. $S = \emptyset$, $M = \overline{P_M}$: The axioms of ACF_T (Definition I.3.1) imply that $[\overline{P_M} : P_M] = \infty$. By [Kei64, Lemma 3.1], there exists some $a \in \overline{P_M}$ such that $[P_M(a) : P_M] > n$.

3. $S \neq \emptyset$: Take some $s_0 \in S$ and define $F = P_M(S \setminus \{s_0\})$. Because M is algebraically closed, there exists an $n+1$-th root $a = s_0^{\frac{1}{n+1}} \in M$. We know that s_0 is transcendental over F, so a is also transcendental over F. Fact I.3.12 implies that $[F(a) : F(s_0)] = n+1$, where $F(s_0) = P_M(S)$ and $F(a) = P_M(S,a)$, as needed.

□

Lemma I.3.14. *Suppose* $\mathrm{trdeg}(M/P_M) \geq \kappa$ *(in particular, if M is κ-saturated) and let $A, A' \subseteq M$ be subsets with $|A|, |A'| < \kappa$. If $f : P_M(A) \to P_M(A')$ is an isomorphism of fields that restricts to an L-automorphism $f|_{P_M}$, then f can be extended to an automorphism of M.*

Proof. From transitivity of transcendental degree

$$\mathrm{trdeg}(M/P_M) = \mathrm{trdeg}(M/P_M(A)) + \mathrm{trdeg}(P_M(A)/P_M),$$

and $\mathrm{trdeg}(P_M(A)/P_M) \leq |A| < \kappa$, so $\mathrm{trdeg}(M/P_M(A)) = \mathrm{trdeg}(M/P_M)$. Similarly, $\mathrm{trdeg}(M/P_M(A')) = \mathrm{trdeg}(M/P_M)$. Let $S, S' \subseteq M$ be transcendence basis of M over $P_M(A), P_M(A')$ respectively, $|S| = \mathrm{trdeg}(M/P_M) = |S'|$. Extend f to an automorphism of fields $\sigma : M \to M$, by mapping $S \mapsto S'$ and extending to the algebraic closure arbitrarily. The restriction $\sigma|_{P_M} = f|_{P_M}$ is an L-automorphism of P, so Lemma I.3.9 implies that σ is an L^P-automorphism. □

I.4 Quantifier elimination and more

I.4.1 Completions

Keisler [Kei64] proved that ACF_T is complete when T is a complete theory in the language of rings. We generalize this by allowing the language of T to be arbitrary.

In his proof, Keisler used special models. We will instead use saturated models, which simplifies the proof, but requires an additional set-theoretic assumption (namely, the generalized continuum hypothesis). There are standard techniques from set theory that ensures the generalized continuum hypothesis from some point on while fixing a fragment of the universe (so this does not affect questions of e.g., completeness of a given theory), see [HK21a], and we will use this freely.

Proposition I.4.1. *If T is a complete theory of fields, then ACF_T is complete.*

Proof. It is enough to show that if $M, N \models \mathrm{ACF}_T$ are saturated models of the same cardinality κ, then they are isomorphic (see the discussion above the proposition). By Lemma I.3.11, $P_M, P_N \models T$ are κ-saturated, and in particular $|P_M| = |P_N| = \kappa$. Because T is complete, [CK90, Theorem 5.1.13] implies that there is an L-isomorphism $\sigma_0 : P_M \to P_N$. By Lemma I.3.13, $\mathrm{trdeg}(M/P_M) = \mathrm{trdeg}(N/P_N) = \kappa$. Let $S \subseteq M$, $S' \subseteq N$ be transcendence basis over P_M, P_N respectively, $|S| = |S'| = \kappa$. We can extend σ_0 to an isomorphism of fields $\sigma_1 : M \to N$, by mapping $S \mapsto S'$ and extending to the algebraic closure arbitrarily. The restriction $\sigma_1|_{P_M}$ is an L-isomorphism, so by Lemma I.3.9 σ_1 is an L^P-isomorphism. □

I.4.2 Quantifier elimination

Our proof of quantifier elimination will be essentially the same as Delon's [Del12, Proposition 14]. One difference is that the criterion used by Delon to prove quantifier elimination assumes a countable language, so we will need a slightly generalized criterion.

In [HKR18], Hils, Kamensky and Rideau proved the same result in a similar fashion. Our proof was derived independently, as we were not aware of their work during the research.

Fact I.4.2. *A theory T has quantifier elimination iff for any two models $M, N \models T$ such that N is $|M|^+$-saturated and any substructures $A \subseteq M$ and $A' \subseteq N$ with an isomorphism $\sigma : A \to A'$, σ can be extended to an embedding $M \to N$.*

Proof. Follows from [Hod93, Theorem 8.4.1]. □

Theorem I.4.3. *If T has quantifier elimination, then ACF^f_T has quantifier elimination.*

Proof. Let $M, N \models \mathrm{ACF}^f_T$ such that N is $|M|^+$-saturated. Let $A \subseteq M$, $A' \subseteq N$ be L^f substructures with isomorphism $\sigma : A \to A'$. By Corollary I.3.8, $\mathrm{Frac}(A) \subseteq M$, $\mathrm{Frac}(A') \subseteq N$ are L^f-substructures with $P_{\mathrm{Frac}(A)} = P_A$, $P_{\mathrm{Frac}(A')} = P_{A'}$. We can extend σ to an isomorphism of fields $\mathrm{Frac}(A) \to \mathrm{Frac}(A')$ that will have the same restriction $P_A \to P_{A'}$, and so by Lemma I.3.9 would still be an L^f-isomorphism. Thus, we can assume without loss of generality that A and A' are subfields. By I.3.11, P_N is $|M|^+$-saturated, and in particular $|P_M|^+$-saturated. The restriction $\sigma|_{P_A} : P_A \to P_{A'}$ is an isomorphism of L-structures from Lemma I.3.9, so quantifier elimination and Fact I.4.2 imply that we can extend $\sigma|_{P_A}$ to an embedding $\sigma_0 : P_M \to P_N$.

Let $B = \sigma_0(P_M) \subseteq P_N$. By Lemma I.3.7, $A \mathop{\smile\!\!\!\!\!|}^l_{P_A} P_M$ and $A' \mathop{\smile\!\!\!\!\!|}^l_{P_{A'}} P_N$, in particular by monotonicity $A' \mathop{\smile\!\!\!\!\!|}^l_{P_{A'}} B$. The field isomorphisms $\sigma : A \to A'$ and $\sigma_0 : P_M \to B$ both restrict to the same isomorphism $P_A \to P_{A'}$, so there is a unique field isomorphism $\sigma_1 : A.P_M \to A'.B$ such that $\sigma_1|_A = \sigma$, $\sigma_1|_{P_M} = \sigma_0$, by Corollary I.2.5.

Let $S \subseteq M$ be a transcendental basis of M over $A.P_M$, $|S| \leq |M|$. From Lemma I.3.13 $\mathrm{trdeg}(N/P_N) \geq |M|^+$ and $|A'| = |A| \leq |M|$, so there exists $S' \subseteq N$ algebraically independent over $A'.P_N$ with $|S| = |S'|$. Let $M' = \overline{A'.B(S')} \subseteq N$. Quantifier elimination implies that the substructure $B \subseteq P_N$ is elementary, so by Corollary I.2.11 $B \subseteq P_N$ is regular. We also know that $A' \mathop{\smile\!\!\!\!\!|}^l_{P_{A'}} P_N$, so by base monotonicity $A'.B \mathop{\smile\!\!\!\!\!|}^l_B P_N$ and by Lemma I.2.9 $\overline{A'.B(S')} \mathop{\smile\!\!\!\!\!|}^l_B P_N$, where $\overline{A'.B(S')} = M'$. Thus, $M' \subseteq N$ is a substructure, with $P_{M'} = B$, from Lemma I.3.7.

We also have $M = \overline{A.P_M(S)}$, so we can extend $\sigma_1 : AP_M \to A'B$ to $\sigma_2 : M \to M'$ by mapping $S \mapsto S'$ arbitrarily and extending to the algebraic closure. In particular, $\sigma_2(P_M) = B = P_{M'}$ and $\sigma_2|_{P_M} = \sigma_0$ is an isomorphism of L-structures, so σ_2 is an isomorphism of L^f-structures by Lemma I.3.9. Thus, σ_2 is an embedding of M into N that extends σ. □

Example I.4.4 ([Del12, Therorem 1])**.** $\mathrm{ACF}^f_{\mathrm{ACF}}$ eliminates quantifiers.

Example I.4.5. $\mathrm{ACF}^f_{\mathrm{RCF}}$ eliminates quantifiers, where RCF is the theory of real closed fields in the language $L_{\mathrm{rings}} \cup \{\leq\}$.

Example I.4.6. Let ACVF be the theory of algebraically closed valued fields in the divisibility language, that is the language of rings with a binary relation $x|y$ signifying $v(x) < v(y)$. ACVF eliminates quantifiers, so $\mathrm{ACF}^f_{\mathrm{ACVF}}$ eliminates quantifiers (by Example I.5.35 it is also NIP).

From quantifier elimination, we can deduce a couple of important corollaries. Both corollaries will rely on expanding a theory T to the Morleyzation, which has quantifier elimination, as defined below.

Definition I.4.7. For a theory T, the *Morleyzation* T_{Mor} of T is an expansion of T by relations $R_\psi(x)$ for any $\psi(x) \in L$, such that $T_{\mathrm{Mor}} \vdash \forall x(R_\psi(x) \leftrightarrow \psi(x))$.

Corollary I.4.8. *Every formula $\phi(x) \in L^P$ is equivalent modulo* ACF_T *to a bounded formula, that is a formula where every quantifier is over P (see Definition I.3.5).*

Proof. Consider the Morleyzation T_{Mor} and the theory $\mathrm{ACF}^f_{T_{\mathrm{Mor}}}$ which has quantifier elimination by Theorem I.4.3. In particular, $\phi(x)$ is equivalent to a quantifier free formula $\phi_0(x) \in L^f_{\mathrm{Mor}}$ modulo $\mathrm{ACF}^f_{T_{\mathrm{Mor}}}$. Replace all occurrences of l_n, $f_{n,i}$ in $\phi_0(x)$ with the formulas defining them, to get an equivalent formula $\phi_1(x) \in L^P_{\mathrm{Mor}}$. The formulas defining l_n, $f_{n;i}$ are bounded, so $\phi_1(x)$ is bounded.

For any formula $\psi(y) \in L$ consider the bounded formula $\psi^P(y) \in L^P$ created from Remark I.3.6. The axioms of $\mathrm{ACF}_{T_{\mathrm{Mor}}}$ (Definition I.3.1) imply that $\mathrm{ACF}_{T_{\mathrm{Mor}}} \vdash \forall y R_\psi(y) \leftrightarrow \psi^P(y)$. Replace each predicate $R_\psi(y)$ in $\phi_1(x)$ by the corresponding $\psi^P(y)$, to get a bounded formula $\phi_2(x) \in L^P$ which is equivalent to $\phi(x)$ modulo ACF_T. □

Remark I.4.9. *In that case that L is the language of rings, Corollary I.4.8 follows from [CZ01, Proposition 2.1], because ACF has* nfcp *and P_M is small in any model $M \models \mathrm{ACF}_T$ (as witnessed in a saturated extension, by Lemma I.3.13).*

Corollary I.4.10. *Let $M, N \models \mathrm{ACF}^f_T$ and let $A \subseteq M$, $B \subseteq N$ be substructures. Then $\sigma : A \to B$ is a partial elementary map from M to N iff $\sigma : A \to B$ is an isomorphism of rings such that $\sigma(P_A) = P_B$ and $\sigma|_{P_A} : P_A \to P_B$ is a partial elementary map from P_M to P_N.*

Proof. Suppose $\sigma : A \to B$ is a partial elementary map from M to N. Then σ is in particular an isomorphism, so $\sigma(P_A) = P_B$. The restriction $\sigma|_{P_A}$ is a partial elementary map from P_M to P_N because for every formula $\phi(x) \in T$, we can apply Remark I.3.6 to get $\phi^P(\bar{x}) \in \mathrm{ACF}_T$, such that $\phi(P_B) = \phi^P(B) = \sigma(\phi^P(A)) = \sigma(\phi(P_A))$.

For the other direction, suppose $\sigma : A \to B$ is an isomorphism of rings such that $\sigma(P_A) = P_B$ and $\sigma|_{P_A} : P_A \to P_B$ is a partial elementary map from P_M to P_N. In particular, $P_M \equiv P_N$, so we can assume that T is the complete theory $T = \mathrm{Th}(P_M) = \mathrm{Th}(P_N)$. Let T_{Mor} be the Morleyzation of T, T_{Mor} has quantifier elimination. We can expand the language of P_M and P_N by definable relations to get $P_M, P_N \models T_{\mathrm{Mor}}$. With this expanded language $M, N \models \mathrm{ACF}^f_{T_{\mathrm{Mor}}}$. The expansion is only relational, so we can still consider A and B as substructure. The restriction $\sigma|_{P_A}$ is a partial elementary map in T, so it is an isomorphism in T_{Mor}, and thus by Lemma I.3.9 σ is an isomorphism in $\mathrm{ACF}^f_{T_{\mathrm{Mor}}}$. By Proposition I.4.1 and Theorem I.4.3 $\mathrm{ACF}^f_{T_{\mathrm{Mor}}}$ is complete and eliminates quantifiers, so σ is a partial elementary map in $\mathrm{ACF}^f_{T_{\mathrm{Mor}}}$. In particular, it is a partial elementary map in ACF^f_T. □

Using this result on elementary maps, we can now show that Lemmas I.3.11 and I.3.13 fully characterize the saturated models of ACF_T.

Proposition I.4.11. *Suppose $\kappa > |L|$, then $N \models \mathrm{ACF}_T$ is κ-saturated iff $P_N \models T$ is κ-saturated and $\mathrm{trdeg}(N/P_N) \geq \kappa$*

Proof. The first direction, if $N \models \mathrm{ACF}_T$ is κ-saturated, then $P_N \models T$ is κ-saturated and $\mathrm{trdeg}(N/P_N) \geq \kappa$, is proved in Lemmas I.3.11 and I.3.13. For the other direction, we will prove κ-homogeneity and κ^+-universality. By expanding the language with definable relations and functions, we can assume $N \models \mathrm{ACF}^f_T$. Let $A, B \subseteq N$ and let $\sigma : A \to B$ be a partial elementary map in N with $\sigma(A) = B$, such that $|A| = |B| < \kappa$. Without loss of generality, we can assume that $A, B \subseteq N$ are L^f-substructures, and by Corollary I.3.8 we can also assume they are subfields. Corollary I.4.10 implies that $\sigma|_{P_A} : P_A \to P_B$ is a partial elementary map in P_N. We know that P_N is κ-homogeneous and $|P_A| = |P_B| < \kappa$, so we can extend $\sigma|_{P_A}$ to an automorphism $\sigma_0 : P_N \to P_N$ in T.

We have $A \mathop{\smile\!\!\!\!\!\mid}^l_{P_A} P_N$ and $B \mathop{\smile\!\!\!\!\!\mid}^l_{P_B} P_N$ from Lemma I.3.7, and the field isomorphisms σ and σ_0 restrict to the same isomorphism $P_A \to P_B$, so by Corollary I.2.5 they can be jointly extended to an isomorphism of fields $\sigma_1 : A.P_N \to B.P_N$. From Lemma I.3.14, σ_1 can be extended to an automorphism of fields $\sigma_2 : N \to N$. Lemma I.3.9 implies that σ_2 is an L^f automorphism because $\sigma_2|_{P_N} = \sigma_0$ is an automorphism in T, and σ_2 extends σ as needed.

Now Let $M \models \mathrm{ACF}_T$ with $|M| \leq \kappa$, by expanding the language we can assume $M \models \mathrm{ACF}^f_T$. We have $P_M \models T$ with $|P_M| < \kappa$, so by κ^+-universality of P_N there exists an elementary embedding $\tau_0 : P_M \to P_N$. Let $B = \tau_0(P_M)$. We have $B \prec P_N$, and in particular from Corollary I.2.11 $B \subseteq P_N$ is a regular extension. Let S be a transcendental basis of M over P_M, $|S| \leq \kappa$ and $\mathrm{trdeg}(N/P_N) \geq \kappa$, so there exists $S_0 \subseteq N$ algebraically independent over P_N with $|S_0| = |S|$. We can extend τ_0 to an embedding $\tau_1 : M \to N$ by mapping $S \mapsto S_0$ arbitrarily and extending to the algebraic closure. Let $M_0 = \tau_1(M) = \overline{B(S_0)}$. From Lemma I.2.9, $\overline{B(S_0)} \mathop{\smile\!\!\!\!\!\mid}^l_B P_N$, so by Lemma I.3.10 $M_0 \subseteq N$ is an L^f-substructure with $P_{M_0} = B$. We have that $\tau_1 : M \to M_0$ is an isomorphism of fields with $\tau_1|_{P_M} = \tau_0 : P_M \to P_{M_0}$ an elementary embedding, so by Corollary I.4.10 τ_1 is an elementary embedding. □

I.4.3 Model completeness

In [Del12, Corollary 15], Delon proved that $\mathrm{ACF}^{ld}_{\mathrm{ACF}}$ is model complete. We can show that if T is model complete, then ACF^{ld}_T is model complete, but in fact we only need a weaker condition — that regular extensions in T are elementary.

Theorem I.4.12. *The following are equivalent:*

1. ACF^f_T *is model complete.*
2. ACF^{ld}_T *is model complete.*
3. *For any $Q, R \models T$ such that $Q \subseteq R$ is a substructure, if $Q \subseteq R$ is a regular extension, then $Q \prec R$.*
4. T_{reg} *(Definition I.2.12) is model complete.*

Proof. 1 $\implies$ 2: Let $M, N \models \mathrm{ACF}^{ld}_T$ with $M \subseteq N$ an L^{ld}-substructure. We can expand M and N uniquely to models of ACF^f_T, by Lemma I.3.10 $M \subseteq N$ is an L^f-substructure. ACF^f_T is model complete, so $M \prec N$ in L^f, in particular $M \prec N$ in L^{ld}.

2 $\implies$ 3: Let $Q, R \models T$ with $Q \subseteq R$ a regular extension. We will construct $M, N \models \mathrm{ACF}^{ld}_T$ such that $P_M = Q$, $P_N = R$ and $M \subseteq N$. We would have liked to take $M = \overline{Q}$, but then we may have $[M : Q] < \infty$, so we should

make M a bit larger. Let s be a new element, transcendental over R. The subfield $Q \subseteq R$ is regular, so by Lemma I.2.9 $\overline{Q(s)} \mathop{\smile\!\!\!\!\!\!|}^{l}_{Q} R$. Define $M = \overline{Q(s)}$, $Q \subseteq M$ is not an algebraic extension so in particular $[M : Q] = \infty$. We have $M \models \mathrm{ACF}^{ld}_{T}$, where we define $P_M = Q$. Similarly, define $N = \overline{R(s)}$, $N \models \mathrm{ACF}^{ld}_{T}$ with $P_N = R$. We know that $P_M \subseteq P_N$ is an L-substructure and $M \mathop{\smile\!\!\!\!\!\!|}^{l}_{P_M} P_N$, so by Lemma I.3.10 $M \subseteq N$ is an L^{ld}-substructure. Model completeness implies $M \prec N$, and in particular $P_M \prec P_N$, because for every formula $\phi(\bar{x}) \in L$ we have $P_M \models \phi(\bar{a}) \iff M \models \phi^P(\bar{a}) \iff N \models \phi^P(\bar{a}) \iff P_N \models \phi(\bar{a})$ for every $\bar{a} \in P_M$, where ϕ^P is given by Remark I.3.6.

3 $\implies$ 1: Let $M, N \models \mathrm{ACF}^{f}_{T}$ and suppose $M \subseteq N$ is a substructure. Lemma I.3.10 implies that $P_M \subseteq P_N$ is an L-substructure and $M \mathop{\smile\!\!\!\!\!\!|}^{l}_{P_M} P_N$. However, M is algebraically closed, so by monotonicity $\overline{P_M} \mathop{\smile\!\!\!\!\!\!|}^{l}_{P_M} P_N$, that is $P_M \subseteq P_N$ is a regular extension. By the assumption, $P_M \prec P_N$. The inclusion map $M \to N$ restricts to the elementary inclusion $P_M \to P_N$, so from Corollary I.4.10 $M \prec N$.

3 $\iff$ 4: Let $Q, R \models T$ be such that $Q \subseteq R$ is an L-extension. By Lemma I.2.13, $Q \subseteq R$ is a regular field extension iff it is an L_{reg}-extension. In particular, regular extensions are elementary iff T_{reg} is model complete. □

Example I.4.13 ([Del12, Corollary 15])**.** $\mathrm{ACF}^{ld}_{\mathrm{ACF}}$ is model complete.

Example I.4.14. $\mathrm{ACF}^{ld}_{\mathrm{PSF}}$ is model complete, where PSF is the theory of pseudo-finite fields in the language of rings (see Proposition I.6.9).

Example I.4.15. $\mathrm{ACF}_{\mathrm{ACF}}$ is *not* model complete. By [TZ12, page 207], the pregeometry of an algebraically closed field K of transcendence degree at least 4 over its prime field with algebraic independence is not modular: there are algebraically closed subfields $A, B \subseteq K$ such that $A \not\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{ACF}}_{A\cap B} B$. Define

$$M = A \qquad N = K$$
$$P_M = A \cap B \qquad P_N = B.$$

It is clear that $M \subseteq N$ is an L^P-substructure, however if $M \prec N$, then Lemma I.3.10 would imply that $A \mathop{\smile\!\!\!\!\!\!|}^{l}_{A\cap B} B$, and in particular $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{ACF}}_{A\cap B} B$, a contradiction.

I.5 Classification and independence

In this section we will assume that T is complete (Proposition I.4.1 implies that ACF_T is also complete) and we will work inside a monster model $\mathbb{M} \models \mathrm{ACF}_T$. Denote $P := P_{\mathbb{M}}$.

Assuming T is NSOP_1, we will define an independence relation $\mathop{\smile\!\!\!\!\!\!|}^{*}$ on $\mathbb{M}$ and prove that it implies Kim-dividing (in fact, Kim^u dividing, see Definition I.2.17) With this result, we will prove that ACF_T is NSOP_1 and that under certain conditions $\mathop{\smile\!\!\!\!\!\!|}^{*}$ is the Kim-independence. We will then expand this result to simplicity and stability.

We will also prove that stability lifts from T to ACF_T using a different approach, by counting types. This approach will let us extend the result to λ-stability.

Finally, we will prove that NIP lifts from T to ACF_T,

I.5.1 Kim-dividing

Definition I.5.1. Call a subfield $A \subseteq \mathbb{M}$ *D-closed* (D for Delon's language) if it is closed under the functions $f_{n,i}$, or equivalently if $A \mathop{\smile\!\!\!\!\!\vert}^{l}_{P_A} P$. For a set $B \subseteq \mathbb{M}$, denote by $\langle B \rangle_D$ the D-closure of B, that is the smallest field containing B and closed under $f_{n,i}$.

Remark I.5.2. *We have the following remarks on D-closure:*

- *In [MPZ20, Definition 3.1], the condition D-closed was called P-special.*
- *If $A \subseteq \mathbb{M}$ is definably closed in L^P, then it is D-closed. In particular, for every $A \subseteq \mathbb{M}$,* $\mathrm{dcl}(A)$ *and* $\mathrm{acl}(A)$ *are D-closed.*
- *D-closure gives a shorter proof of local character of $\mathop{\smile\!\!\!\!\!\vert}^{l}$ (see Fact I.2.4). Suppose a is finite and B is an infinite field and consider the structure $(B(a), B)$ (the field $B(a)$ with a predicate for B). Define $B_0 = \langle a \rangle_D \cap B$, which is countable. We have $B_0(a) = \langle a \rangle_D$, so $B_0(a) \mathop{\smile\!\!\!\!\!\vert}^{l}_{B_0} B$.*

Lemma I.5.3. *Suppose $A, B, C \subseteq \mathbb{M}$ are subfields with $C \subseteq A \cap B$. If A is D-closed, then $A.P \mathop{\smile\!\!\!\!\!\vert}^{l}_{C.P} B.P$ iff $A \mathop{\smile\!\!\!\!\!\vert}^{l}_{C.P_A} B.P$. By symmetry, if B is D-closed, then $A.P \mathop{\smile\!\!\!\!\!\vert}^{l}_{C.P} B.P$ iff $A.P \mathop{\smile\!\!\!\!\!\vert}^{l}_{C.P_B} B$. Furthermore, if both A and B are D-closed, then $A.P \mathop{\smile\!\!\!\!\!\vert}^{l}_{C.P} B.P$ implies $A.B \mathop{\smile\!\!\!\!\!\vert}^{l}_{P_A.P_B} P$, i.e. $P_{A.B} = P_A.P_B$ and $A.B$ is D-closed.*

Proof. If $A \mathop{\smile\!\!\!\!\!\vert}^{l}_{C.P_A} B.P$, then $A.P \mathop{\smile\!\!\!\!\!\vert}^{l}_{C.P} B.P$ from base monotonicity. On the other hand, if $A.P \mathop{\smile\!\!\!\!\!\vert}^{l}_{C.P} B.P$, then because $A \mathop{\smile\!\!\!\!\!\vert}^{l}_{P_A} P$ implies $A \mathop{\smile\!\!\!\!\!\vert}^{l}_{C.P_A} C.P$ from base monotonicity, we get from transitivity that $A \mathop{\smile\!\!\!\!\!\vert}^{l}_{C.P_A} B.P$. For the furthermore part, we know from $A \mathop{\smile\!\!\!\!\!\vert}^{l}_{P_A} P$ and $A.P \mathop{\smile\!\!\!\!\!\vert}^{l}_{C.P} B.P$ that $A \mathop{\smile\!\!\!\!\!\vert}^{l}_{C.P_A} B.P$. By base monotonicity, $A.B \mathop{\smile\!\!\!\!\!\vert}^{l}_{B.P_A} B.P$. Also, from $B \mathop{\smile\!\!\!\!\!\vert}^{l}_{P_B} P$ and base monotonicity, $B.P_A \mathop{\smile\!\!\!\!\!\vert}^{l}_{P_A.P_B} P$, thus by transitivity $A.B \mathop{\smile\!\!\!\!\!\vert}^{l}_{P_A.P_B} P$. ☐

Definition I.5.4. Let $M \prec \mathbb{M}$ and $A, B \subseteq \mathbb{M}$ be small D-closed subfields, such that $M \subseteq A \cap B$. Define $A \mathop{\smile\!\!\!\!\!\vert}^{*}_{M} B$ if

1. $P_A \mathop{\smile\!\!\!\!\!\vert}^{K}_{P_M} P_B$ in P.
2. $A.P \mathop{\smile\!\!\!\!\!\vert}^{l}_{M.P} B.P$.

Lemma I.5.5. *Let $A, B, C \subseteq \mathbb{M}$ be small subsets with $C \subseteq A \cap B$. If $A \mathop{\smile\!\!\!\!\!\vert}^{u}_{C} B$, then:*

1. *$P_A \mathop{\smile\!\!\!\!\!\vert}^{u}_{P_C} P_B$ in P.*
2. *If A, B and C are subfields and B is D-closed, then $A.P \mathop{\smile\!\!\!\!\!\vert}^{l}_{C.P} B.P$.*

In particular, if $M \prec \mathbb{M}$ and A and B are D-closed with $M \subseteq A \cap B$, then $A \mathop{\smile\!\!\!\!\!\vert}^{u}_{M} B$ implies $A \mathop{\smile\!\!\!\!\!\vert}^{*}_{M} B$.

Proof. For point (1), suppose $P \models \phi(a, b)$ for some formula $\phi(x, y) \in L$, $a \in P_A$ and $b \in P_B$. Let $\phi^P(x, y) \in L^P$ be as in Remark I.3.6, we have $\mathbb{M} \models \phi^P(a, b)$. By $A \mathop{\smile\!\!\!\!\!\!|}^u_C B$ there is some $c \in C$ such that $\mathbb{M} \models \phi^P(c, b)$. Thus, $c \in P \cap C = P_C$, and we have $P \models \phi(c, b)$.

For point (2), it is enough to prove $A.P \mathop{\smile\!\!\!\!\!\!|}^l_{C.P_B} B$ by Lemma I.5.3. Let $\sum_i u_i b_i = 0$ for $u_i \in A.P$, $b_i \in B$ such that the u_i are not all trivial (not all equal to 0). We can write $u_i = f_i(\bar{a}_i, \bar{p}_i)$ for $f_i \in C(\bar{x}_i, \bar{y}_i)$ a rational function, $\bar{a}_i \in A$, $\bar{p}_i \in P$. Thus, we have $\models \sum_i f_i(\bar{a}_i, \bar{p}_i)b_i = 0 \wedge \bigvee_i f_i(\bar{a}_i, \bar{p}_i) \neq 0$, and in particular $\models \exists \bar{y}_i \in P, \sum_i f_i(\bar{a}_i, \bar{y}_i)b_i = 0 \wedge \bigvee_i f_i(\bar{a}_i, \bar{y}_i) \neq 0$. From $A \mathop{\smile\!\!\!\!\!\!|}^u_C B$, there are $\bar{c}_i \in C$ such that $\models \exists \bar{y}_i \in P \sum_i f_i(\bar{c}_i, \bar{y}_i)b_i = 0 \wedge \bigvee_i f_i(\bar{c}_i, \bar{y}_i) \neq 0$. Let $\bar{q}_i \in P$ witness the existence, and let $v_i = f_i(\bar{c}_i, \bar{q}_i) \in C.P$. We have $\sum_i v_i b_i = 0$ and v_i are not all trivial. Moreover, $B \mathop{\smile\!\!\!\!\!\!|}^l_{P_B} P$, so by base monotonicity $B \mathop{\smile\!\!\!\!\!\!|}^l_{C.P_B} C.P$, thus there are $w_i \in C.P_B$, not all trivial, such that $\sum_i w_i b_i = 0$, as needed.

The "in particular" part follows from the definition of $\mathop{\smile\!\!\!\!\!\!|}^*$, because $P_A \mathop{\smile\!\!\!\!\!\!|}^u_{P_M} P_B$ implies $P_A \mathop{\smile\!\!\!\!\!\!|}^K_{P_M} P_B$ (see [dE21a, Fact 3.10]). □

Lemma I.5.6. *Let $A, B, C \subseteq \mathbb{M}$ be small subsets with $C \subseteq A \cap B$ and let $(B_i)_{i<\omega}$ be a C-indiscernible coheir sequence such that $B \equiv_A B_i$, then $(P_{B_i})_{i<\omega}$ is a P_C-indiscernible coheir sequence such that $P_B \equiv_{P_A} P_{B_i}$ in P.*

Proof. For every formula in P, we can restrict all quantifiers and free variables to be over P to get a formula in $\mathbb{M}$ with the same definable set. This proves that $(P_{B_i})_{i<\omega}$ is P_C-indiscernible and $P_B \equiv_{P_A} P_{B_i}$ in P. From Lemma I.5.5, $P_{B_i} \mathop{\smile\!\!\!\!\!\!|}^u_{P_C} P_{B_{<i}}$ in P, and $P_{B_{<i}} = \bigcup_{j<i} P_{B_j}$, so $(P_{B_i})_{i<\omega}$ is a P_C-indiscernible coheir sequence. □

Proposition I.5.7. *Assume T is* NSOP_1. *Let $M \prec \mathbb{M}$ and let $A, B \subseteq \mathbb{M}$ be small D-closed subfields with $M \subseteq A \cap B$, such that A is algebraically closed as a field. If $A \mathop{\smile\!\!\!\!\!\!|}^*_M B$, then* $\mathrm{tp}(A/B)$ *does not Kim^u-divide over M (recall Definition I.2.17).*

Proof. Let $(B_i)_{i<\omega}$ be any M-indiscernible coheir sequence such that $B \equiv_M B_i$ for every $i < \omega$ and let $\beta_i : B \to B_i$ be L^P-isomorphisms matching the tuples. By Lemma I.5.6, $(P_{B_i})_{i<\omega}$ is a P_M-indiscernible coheir sequence. Because T is NSOP_1 and $P_A \mathop{\smile\!\!\!\!\!\!|}^K_{P_M} P_B$, Fact I.2.20(2) implies that there exists $Q \subseteq P$ such that $P_A P_B \equiv_{P_M} Q P_{B_i}$ in P for all $i < \omega$. This means that there are automorphisms γ_i of P mapping $P_A P_B$ to $Q P_{B_i}$ and preserving P_M pointwise, such that the restriction $\gamma_i|_{P_A} : P_A \to Q$ is the same for every $i < \omega$, call it $\alpha_0 : P_A \to Q$.

Let $S \subseteq A$ be a transcendence basis of A over $M.P_A$. By Lemma I.3.13, $\mathrm{trdeg}(\mathbb{M}/P) = |\mathbb{M}|$, so there exists some S' algebraically independent over $B_{<\omega}P$ with $|S'| = |S|$. Define $A' = \overline{M.Q(S')}$. From Lemma I.3.7, $M \mathop{\smile\!\!\!\!\!\!|}^l_{P_M} P$, so from monotonicity $M \mathop{\smile\!\!\!\!\!\!|}^l_{P_M} P_A$ and $M \mathop{\smile\!\!\!\!\!\!|}^l_{P_M} Q$. Thus, from stationarity of $\mathop{\smile\!\!\!\!\!\!|}^l$, we can extend $\alpha_0 : P_A \to Q$ to an isomorphism of fields $M.P_A \to M.Q$ preserving M pointwise. Map $S \mapsto S'$ arbitrarily and extend arbitrarily to the algebraic closure, to get an isomorphism of fields $\alpha : A \to A'$. This give us a way to consider A' as a tuple.

Let $i < \omega$. We know that $B \mathop{\smile\!\!\!\!\!\!|}^l_{P_B} P$ and $B_i \mathop{\smile\!\!\!\!\!\!|}^l_{P_{B_i}} P$, the field isomorphisms $\beta_i : B \to B_i$ and $\gamma_i : P \to P$ both restrict to the same isomorphism $P_B \to$

P_{B_i}, so from Corollary I.2.5 they can be jointly extended to an isomorphism of fields $\sigma_{i,0} : B.P \to B_i.P$. From $A.P \mathop{\smile\!\!\!\!\!\mid}\,^l_{M.P} B.P$ and Lemma I.5.3 we get that $A \mathop{\smile\!\!\!\!\!\mid}\,^l_{M.P_A} B.P$. We would like to prove that also $A' \mathop{\smile\!\!\!\!\!\mid}\,^l_{M.Q} B_i.P$. We know that A is algebraically closed, so $M.P_A \subseteq B.P$ is regular. Applying Lemma I.2.8 with $\sigma_{i,0}$, we get that $M.Q \subseteq B_i.P$ is regular. The set S' is algebraically independent over $B_i.P$, so from Lemma I.2.9 $\overline{M.Q(S')} \mathop{\smile\!\!\!\!\!\mid}\,^l_{M.Q} B_i.P$, where $\overline{M.Q(S')} = A'$.

The isomorphisms of fields $\alpha : A \to A'$ and $\sigma_{i,0} : B.P \to B_i.P$ restrict to the same isomorphism $M.P_A \to M.Q$, which acts as α_0 on P_A and preserves M pointwise. Thus, from Corollary I.2.5, they can be jointly extended to an isomorphism of fields $\sigma_{i,1} : A.B.P \to A'.B_i.P$. By Lemma I.3.14, $\sigma_{i,1}$ can be extended to $\sigma_{i,2}$ an L^P-automorphism of $\mathbb{M}$. The automorphism $\sigma_{i,2}$ maps $AB \mapsto A'B_i$ and preserves M pointwise, so $AB \equiv_M A'B_i$. This is compatible with the way we defined A' as a tuple, because $\sigma_{i,2}|_A = \alpha$. □

I.5.2 NSOP$_1$, simplicity

Remark I.5.8. *In a general theory T, if $A \mathop{\smile\!\!\!\!\!\mid}\,^u_C B$, then $\operatorname{acl}(AC) \mathop{\smile\!\!\!\!\!\mid}\,^u_{\operatorname{acl}(C)} \operatorname{acl}(BC)$. Indeed, by extension, for some $A' \equiv_{BC} A$ we have $A' \mathop{\smile\!\!\!\!\!\mid}\,^u_C \operatorname{acl}(BC)$, and by applying an automorphism taking A' to A and fixing BC we get that $A \mathop{\smile\!\!\!\!\!\mid}\,^u_C \operatorname{acl}(BC)$. By base monotonicity, $A \mathop{\smile\!\!\!\!\!\mid}\,^u_{\operatorname{acl}(C)} \operatorname{acl}(BC)$.*

Suppose that $\models \phi(d, b)$ where $\phi(x, y)$ is a formula over $\operatorname{acl}(C)$, $d \in \operatorname{acl}(AC)$ and $b \in \operatorname{acl}(BC)$. Let $\psi(x, z)$ be a formula over C and $a \in A$ be such that $\psi(x, a)$ is algebraic, say of size n, and $\models \psi(d, a)$. By the first part there exist $c \in \operatorname{acl}(C)$ such that $\psi(x, c)$ is of size n and $\models \exists x(\phi(x, b) \land \psi(x, c))$, let e witness the existence. The fact that $\models \psi(e, c)$ implies that $e \in \operatorname{acl}(C)$, and we have $\models \phi(e, b)$, so $\operatorname{acl}(AC) \mathop{\smile\!\!\!\!\!\mid}\,^u_{\operatorname{acl}(C)} \operatorname{acl}(BC)$.

Theorem I.5.9. *If T is* NSOP$_1$*, then* ACF$_T$ *is* NSOP$_1$*.*

Proof. We will use Fact I.2.16. Let $M \prec \mathbb{M}$ and suppose A_0, A_1, B_0 and B_1 are such that $A_0B_0 \equiv_M A_1B_1$, $B_1 \mathop{\smile\!\!\!\!\!\mid}\,^u_M B_0$ and $B_i \mathop{\smile\!\!\!\!\!\mid}\,^u_M A_i$ for $i = 0, 1$. By Remark I.5.8, we can assume that $A_i = \operatorname{acl}(A_iM)$, $B_i = \operatorname{acl}(B_iM)$, and in particular they are all D-closed and algebraically closed.

From $B_0 \mathop{\smile\!\!\!\!\!\mid}\,^u_M A_0$, we get using Lemma I.5.5 that $B_0 \mathop{\smile\!\!\!\!\!\mid}\,^*_M A_0$. However, T is NSOP$_1$, so Fact I.2.20(3) implies that $\mathop{\smile\!\!\!\!\!\mid}\,^K$ in P is symmetric, thus $\mathop{\smile\!\!\!\!\!\mid}\,^*$ is also symmetric and we have $A_0 \mathop{\smile\!\!\!\!\!\mid}\,^*_M B_0$. By Proposition I.5.7, $\operatorname{tp}(A_0/B_0)$ does not Kim^u-divide over M. Extend the pair (B_0, B_1) to a coheir sequence $(B_i)_{i<\omega}$ (to do that, first extend $\operatorname{tp}(B_1/MB_0)$ to a global type which is finitely satisfiable in M, and then generate a Morley sequence in that type; see [KR20, §3.1]). By the definition of Kim^u-dividing (Definition I.2.17) we get that there exists $A \subseteq \mathbb{M}$ such that $A_0B_0 \equiv_M AB_0 \equiv_M AB_1$. □

Example I.5.10. The theory of ω-free PAC fields was shown to be non-simple by Chatzidakis [Cha99], as it is PAC and unbounded, and NSOP$_1$ by Chernikov and Ramsey [CR16]. Thus, $ACF_{\omega\text{-free PAC}}$ is NSOP$_1$ and non-simple as the theory of ω-free PAC fields is interpretable in $ACF_{\omega\text{-free PAC}}$.

Now we will show that in NSOP$_1$ theories, Kim-independence is $\mathop{\smile\!\!\!\!\!\mid}\,^*$ for certain sets.

Proposition I.5.11. *Assume T is* NSOP_1*. Let $M \prec \mathbb{M}$ and let $A, B \subseteq \mathbb{M}$ be small D-closed subfields with $M \subseteq A \cap B$. Then $A \mathop{\smile\!\!\!\!\!\!\mid}^K_M B$ implies $A \mathop{\smile\!\!\!\!\!\!\mid}^*_M B$. If either A or B are algebraically closed as fields, then also $A \mathop{\smile\!\!\!\!\!\!\mid}^*_M B$ implies $A \mathop{\smile\!\!\!\!\!\!\mid}^K_M B$.*

Proof. The theory T is NSOP_1, so ACF_T is also NSOP_1 from Theorem I.5.9. Suppose $A \mathop{\smile\!\!\!\!\!\!\mid}^K_M B$, we need to prove that $P_A \mathop{\smile\!\!\!\!\!\!\mid}^K_{P_M} P_B$ in P and $A.P \mathop{\smile\!\!\!\!\!\!\mid}^l_{M.P} B.P$. There exists an M-indiscernible coheir sequence $(B_i)_{i<\omega}$, with $B \equiv_A B_i$. From Lemma I.5.6, $(P_{B_i})_{i<\omega}$ is a P_M indiscernible coheir sequence with $P_B \equiv_{P_A} P_{B_i}$ in P. Because T is NSOP_1, Fact I.2.20(1) implies that $P_A \mathop{\smile\!\!\!\!\!\!\mid}^K_{P_M} P_B$.

To prove that $A.P \mathop{\smile\!\!\!\!\!\!\mid}^l_{M.P} B.P$, it is enough to prove that $A \mathop{\smile\!\!\!\!\!\!\mid}^l_{M.P_A} B.P$, by Lemma I.5.3. Let $\overline{a} \in A$ be a finite tuple and suppose it is linearly dependent over $B.P$. Because $A \mathop{\smile\!\!\!\!\!\!\mid}^K_M B$, we can construct an uncountable M-indiscernible coheir sequence $(B_i)_{i<\omega_1}$, with $B \equiv_A B_i$. Let $\sigma_i \in \mathrm{Aut}(\mathbb{M}/A)$ be an automorphism mapping B to B_i. We know that σ_i preserves P setwise, so by applying σ_i we get that $\overline{a}$ is linearly dependent over $B_i.P$. By local character, there is some countable subfield $C \subseteq \mathrm{acl}(B_{<\omega_1}).P$ such that $C(\overline{a}) \mathop{\smile\!\!\!\!\!\!\mid}^l_C \mathrm{acl}(B_{<\omega_1}).P$. Because C is countable, there is some $i < \omega_1$ such that $C \subseteq \mathrm{acl}(B_{<i}).P$. By Remark I.5.8 we have $B_i \mathop{\smile\!\!\!\!\!\!\mid}^u_M \mathrm{acl}(B_{<i})$, so Lemma I.5.5 implies that $B_i.P \mathop{\smile\!\!\!\!\!\!\mid}^l_{M.P} \mathrm{acl}(B_{<i}).P$, and in particular from monotonicity $B_i.P \mathop{\smile\!\!\!\!\!\!\mid}^l_{M.P} M.P.C$. However, the fact that $C(\overline{a}) \mathop{\smile\!\!\!\!\!\!\mid}^l_C \mathrm{acl}(B_{<\omega_1}).P$ also implies, using monotonicity, base monotonicity and symmetry, that $B_i.P.C \mathop{\smile\!\!\!\!\!\!\mid}^l_{M.P.C} M.P.C(\overline{a})$, so by transitivity $B_i.P \mathop{\smile\!\!\!\!\!\!\mid}^l_{M.P} M.P.C(\overline{a})$. The tuple $\overline{a}$ is linearly dependent over $B_i.P$, so it is linearly dependent over $M.P$. However, A is D-closed so $A \mathop{\smile\!\!\!\!\!\!\mid}^l_{P_A} P$ and by base monotonicity $A \mathop{\smile\!\!\!\!\!\!\mid}^l_{M.P_A} M.P$. Thus, $\overline{a}$ is linearly dependent over $M.P_A$, as needed.

If A is algebraically closed and $A \mathop{\smile\!\!\!\!\!\!\mid}^*_M B$, then from Proposition I.5.7 $\mathrm{tp}(A/B)$ does not Kim^u-divide over M. ACF_T is NSOP_1, so by Remark I.2.18 Kim^u-dividing is the same as Kim-dividing, and by Fact I.2.20(2) Kim-dividing is the same as Kim-forking, thus $A \mathop{\smile\!\!\!\!\!\!\mid}^K_M B$. The case where B is algebraically closed follows from symmetry of $\mathop{\smile\!\!\!\!\!\!\mid}^*$ and $\mathop{\smile\!\!\!\!\!\!\mid}^K$ (Fact I.2.20(3)). ☐

Remark I.5.12. *The proof of Proposition I.5.11 was inspired by the proof of [BYPV03, Proposition 7.3]*

Theorem I.5.13. *If T is simple, then* ACF_T *is simple.*

Proof. Suppose T is simple, in particular T is NSOP_1 so Theorem I.5.9 implies that ACF_T is NSOP_1. By Fact I.2.20(5), for an NSOP_1 theory being simple is equivalent to Kim-independence having base monotonicity. Let $A, B \subseteq \mathbb{M}$ be small subsets and $M, N \prec \mathbb{M}$ submodels, such that $M \subseteq A$, $M \subseteq N \subseteq B$. Suppose $A \mathop{\smile\!\!\!\!\!\!\mid}^K_M B$, we want to prove $A \mathop{\smile\!\!\!\!\!\!\mid}^K_N B$. Without loss of generality we can assume that A and B are acl-closed.

By Proposition I.5.11, $A \mathop{\smile\!\!\!\!\!\!\mid}^K_M B$ implies $A \mathop{\smile\!\!\!\!\!\!\mid}^*_M B$. We have $A.P \mathop{\smile\!\!\!\!\!\!\mid}^l_{M.P} B.P$, and by monotonicity $A.P \mathop{\smile\!\!\!\!\!\!\mid}^l_{M.P} N.P$, so from Lemma I.5.3 $N.A$ is D-closed. Since B is D-closed and algebraically closed as a field, by Proposition I.5.11 it is enough to prove $N.A \mathop{\smile\!\!\!\!\!\!\mid}^*_N B$. By base monotonicity of linear disjointness, $A.P \mathop{\smile\!\!\!\!\!\!\mid}^l_{M.P} B.P$ implies $N.A.P \mathop{\smile\!\!\!\!\!\!\mid}^l_{N.P} B.P$. We know that T is simple, so by base monotonicity of Kim-independence in P, $P_A \mathop{\smile\!\!\!\!\!\!\vert}^K_{P_M} P_B$ implies $P_N.P_A \mathop{\smile\!\!\!\!\!\!\vert}^K_{P_N} P_B$. ☐

Example I.5.14. $\mathrm{ACF}_{\mathrm{PSF}}$ is simple, where PSF is the theory of pseudo-finite fields (see Proposition I.6.10 for an alternative proof).

I.5.3 Stability

There are a few ways to prove that if T is stable, then ACF_T is stable. The first option, continuing in the path of the previous results, is using a Kim-Pillay style characterization on non-forking independence, which in simple theories is the same as Kim-independence over models.

The second option is a more direct approach, by counting types. The second option will give us a stronger result, that if T is λ-stable, then so is ACF_T, which will let us extend to super-stability and ω-stability. Even though the second option is strictly stronger than the first, we will also show the first, to complete the picture on Kim-independence.

A third way to prove stability, is by proving the existence of saturated models of certain cardinalities. This could be done using the characterization of saturated models of ACF_T found in Proposition I.4.11, but we will not expand on it here.

Remark I.5.15. *When the predicate has no extra structure, stability can also be deduced from [CZ01, Corollary 5.4] (which cites [Pil98], probably meaning Proposition 3.1 there), which is a much more general statement: if M is strongly minimal and A is some subset of M such that the induced structure on A is stable, then (M, A) is stable.*

Theorem I.5.16. *If T is stable, then* ACF_T *is stable.*

Proof. Suppose T is stable, in particular T is simple so Theorem I.5.13 implies that ACF_T is simple. [KR20, Proposition 8.4] says that in simple theories, non-forking independence over models is the same as Kim-independence. To show that ACF_T is stable, it is enough to show that non-forking independence has stationarity over models ([Cas11, Theorem 12.22]). Let A, A' and B be small subsets such that $M \subseteq A \cap A' \cap B$. Suppose $A \mathop{\smile\!\!\!\!\!\!\vert}^K_M B$, $A' \mathop{\smile\!\!\!\!\!\!\vert}^K_M B$ and $A \equiv_M A'$. Without loss of generality we can assume A, A' and B are acl-closed.

We have $P_A \equiv_{P_M} P_{A'}$, and by Proposition I.5.11 $P_A \mathop{\smile\!\!\!\!\!\!\vert}^K_{P_M} P_B$ and $P_{A'} \mathop{\smile\!\!\!\!\!\!\vert}^K_{P_M} P_B$ in P. We know that T is stable, so by stationarity $P_A \equiv_{P_B} P_{A'}$. Let σ_0 be an automorphism of P mapping P_A to $P_{A'}$ and preserving P_B pointwise. We have $B \mathop{\smile\!\!\!\!\!\!\vert}^l_{P_B} P$, so by stationarity of linear disjointedness we can extend σ_0 to $\sigma_1 : B.P \to B.P$ preserving B pointwise. By Proposition I.5.11 and Lemma I.5.3, $A \mathop{\smile\!\!\!\!\!\!\vert}^l_{M.P_A} B.P$ and $A' \mathop{\smile\!\!\!\!\!\!\vert}^l_{M.P_{A'}} B.P$, so by Corollary I.2.5 we can extend σ_1 to $\sigma_2 : A.B.P \to A'.B.P$, mapping A to A'. Extend σ_2 to σ_3, an automorphism of $\mathbb{M}$, using Lemma I.3.14. The automorphism σ_3 maps A to A' and preserves B pointwise, so $A \equiv_B A'$. ☐

Example I.5.17. $\mathrm{ACF}_{\mathrm{SCF}}$ is stable, where SCF is the theory of separably closed fields.

To prove stability by counting types, we will need to show that P is stably embedded in $\mathbb{M}$.

Definition I.5.18. A set $Q \subseteq \mathbb{M}^m$ which is definable over the empty set is called *stably embedded* if for every n, if $D \subseteq \mathbb{M}^{mn}$ is definable, then $D \cap Q^n$ is definable with parameters from Q.

Fact I.5.19 ([Cha99, Appendix, Lemma 1])**.** *For $Q \subseteq \mathbb{M}$ as above, if every automorphism of the induced structure on Q lifts to an automorphism of $\mathbb{M}$, then Q is stably embedded.*

Remark I.5.20. *The precise formulation of the above fact is more general but requires extra assumptions on T, namely that $T = T^{eq}$ and that the language is countable. However, those assumptions are not used in the proof of the direction we cited.*

Lemma I.5.21. *The induced structure on P as a subset of $\mathbb{M}$ is the same (up to inter-definability) as the intrinsic L-structure of P.*

Proof. If $A \subseteq P^n$ is definable in P by a formula $\phi \in L$, then we can construct by Remark I.3.6 a formula $\phi^P \in L^P$ that defines A in $\mathbb{M}$.

In the other direction, if $A \subseteq P^n$ is definable in $\mathbb{M}$ by a formula $\psi \in L^P$, then we can assume by Corollary I.4.8 that ψ is bounded. Remove any occurrence of P in ψ, by replacing $x \in P$ with a tautology $(x = x)$, to get a formula in L that defines A in P. □

Proposition I.5.22. *P is stably embedded in $\mathbb{M}$.*

Proof. Follows from Fact I.5.19 and Lemmas I.3.14 and I.5.21. □

Remark I.5.23. *It follows from a simple compactness argument that P is even* uniformly *stably embedded, that is, for any formula $\phi(x, y)$ there exists a formula $\psi(x, z)$ such that for every $b \in \mathbb{M}$ there is $c \in P$ with $\phi(P, b) = \psi(P, c)$.*

Theorem I.5.24. *If T is λ-stable, then ACF_T is λ-stable.*

Proof. Suppose T is λ-stable, we can assume that $|T| \leq \lambda$ by replacing T up to inter-definability (see e.g. [TZ12, Exercise 5.2.6]). Let $C \subseteq \mathbb{M}$ be a subset with $|C| \leq \lambda$, we need to prove that $|S_1^{\mathrm{ACF}_T}(C)| \leq \lambda$, where $S_1^{\mathrm{ACF}_T}(C)$ is the space of types in one variable over C. First we will prove that all elements in $\mathbb{M} \setminus \overline{P(C)}$ have the same type over C. Suppose $a_0, a_1 \in \mathbb{M} \setminus \overline{P(C)}$, that is both a_0 and a_1 are transcendental over $P(C)$. There is an isomorphism of fields $P(C, a_0) \to P(C, a_1)$ given by fixing $P(C)$ pointwise and mapping $a_0 \mapsto a_1$. By Lemma I.3.14, we can extend this map to an automorphism of $\mathbb{M}$, so $a_0 \equiv_C a_1$.

It remains to show that there are at most λ types in $\overline{P(C)}$. Any element of $\overline{P(C)}$ solves some non-zero polynomial of the form $q(x; b, c)$ with $b \in P^n$ and $c \in C^m$, and in particular satisfies

$$\phi(x; c) = \exists y \in P \ (q(x; y, c) = 0 \wedge \exists x' q(x'; y, c) \neq 0).$$

Thus, any type in $\overline{P(C)}$ contains some formula $\phi(x; c)$ as above. There are at most λ formulas in L^P with parameters from C, so it is enough to prove that there are at most λ types that contain any given formula $\phi(x; c)$ as above.

First of all, P is stably embedded in $\mathbb{M}$ (Proposition I.5.22), so every C-definable subset of P^n is also definable in ACF_T with parameters from P. Let $D \subseteq P$ be the set of all the parameters needed to define every C-definable subset of P^n. There are at most λ definable subsets of P^n over C, so $|D| \leq \lambda$.

Let $[\phi] \subseteq S_1^{\mathrm{ACF}_T}(C)$ be the set of types implying $\phi(x;c)$. We will construct a map $\rho : [\phi] \to S_n^T(D)$ such that ρ has finite fibers. Because T is λ-stable, $|S_n^T(D)| \leq \lambda$, so this will imply $|[\phi]| \leq \lambda$ as needed.

For any type $p(x) \in [\phi]$, choose some realization $a \models p$. In particular, $\models \phi(a;c)$, so we can choose some $b \in P^n$ such that $q(x;b,c)$ is non-zero and $q(a;b,c) = 0$. Define $\rho(p) = \mathrm{tp}^T(b/D)$. Suppose $p_0, p_1 \in [\phi]$ and $\rho(p_0) = \rho(p_1)$, that is, if a_i, b_i are the specific elements we chose for p_i $(i = 0, 1)$, then $b_0 \equiv_D^T b_1$. There is an automorphism of P over D mapping $b_0 \mapsto b_1$, which can be extended by Lemma I.3.14 to an automorphism of $\mathbb{M}$ over D, so $b_0 \equiv_D^{\mathrm{ACF}_T} b_1$. We want to prove that $b_0 \equiv_C^{\mathrm{ACF}_T} b_1$. Suppose b_0 belongs to some C-definable set, we can assume that it is a subset of P^n because $b_0 \in P^n$. By the construction of D, this C-definable subset of P^n is also D-definable in ACF_T, so b_1 belongs to it as $b_0 \equiv_D^{\mathrm{ACF}_T} b_1$.

Let $\sigma \in \mathrm{Aut}(\mathbb{M}/C)$ be an automorphism mapping b_0 to b_1. We have $q(\sigma(a_0); b_1, c) = 0$, thus a_0 has the same type over C as a root of $q(x; b_1, c)$, specifically $\sigma(a_0)$. It follows that every type in the fiber of $\rho(p_1)$ is a type over C of a root of $q(x; b_1, c)$, however $q(x; b_1, c)$ is non-zero, so it has only finitely many roots. Thus, ρ has finite fibers. □

We can apply Theorem I.5.24 on a specific λ to give another proof of Theorem I.5.16. We also get the following corollaries:

Corollary I.5.25. *If T is superstable, then ACF_T is superstable.*

Corollary I.5.26. *If T is ω-stable, then ACF_T is ω-stable.*

Example I.5.27. $\mathrm{ACF}_{\mathrm{ACF}}$ is ω-stable, see Proposition I.6.2 for an extended application of this result.

Remark I.5.28. *Per the definition in [Poi83],* $\mathrm{ACF}_{\mathrm{ACF}}$ *is a* belle pair*, so it is stable. In [BYPV03], the notion of belle pairs was expanded to* lovely pairs *and a description of non-forking independence was given. When considering pairs of ACF, the description of non-forking independence in Proposition I.5.11 is slightly different from the description given in [BYPV03, Proposition 7.3] — instead of the condition $A.P \mathop{\smile\!\!\!\!\!\mid}^{l}_{M.P} B.P$ they have $A.P \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{ACF}}_{M.P} B.P$. However, in this case the conditions are equivalent, as can be seen in [MPZ20, Corollary 6.2].*

I.5.4 NIP

We will prove that if T is NIP, then ACF_T is NIP. First we will define the notions of a NIP formula, type and theory, and present some basic facts based on [Sim15] and [KS14].

Definition I.5.29. A formula $\phi(x,y)$ has the *independence property* (IP) if there is a sequence $(a_i)_{i<\omega}$ such that for every $s \subseteq \omega$ the set $\{\phi(a_i, y) \mid i \in s\} \cup \{\neg\phi(a_i, y) \mid i \notin s\}$ is consistent.

A partial type $\pi(x)$ has IP if there is a formula $\phi(x,y)$ and a sequence $(a_i)_{i<\omega}$ of realizations $a_i \models \pi(x)$ such that for every $s \subseteq \omega$ the set $\{\phi(a_i, y) \mid i \in s\} \cup \{\neg\phi(a_i, y) \mid i \notin s\}$ is consistent. Otherwise, $\pi(x)$ is NIP.

A theory T has IP if some formula has IP modulo T, or equivalently the type $x = x$ has IP. Otherwise, T is NIP.

Fact I.5.30 ([Sim15, Lemma 2.7]). *A formula $\phi(x, y)$ has IP iff there is an indiscernible sequence $(a_i)_{i<\omega}$ and a tuple b such that $\models \phi(a_i, b) \iff i$ is even.*

Fact I.5.31 ([Sim15, Proposition 2.11]). *A theory T is NIP iff no formula $\phi(x, y)$ with $|y| = 1$ has IP modulo T.*

Fact I.5.32 ([KS14, Proposition 2.6]). *Suppose $\pi(x)$ is a partial NIP type over A and B is a set of realizations of $\pi(x)$. If $I = (a_i)_{i<|T|^+ + |B|^+}$ is an A-indiscernible sequence, then some end segment of I is indiscernible over AB.*

First we need to show that P is NIP per Definition I.5.29

Lemma I.5.33. *If T is NIP, then P is NIP, i.e. the partial type $x \in P$ is NIP.*

Proof. Suppose $x \in P$ has IP. Then there are a sequence $(a_i)_{i<\omega}$ with $a_i \in P$ and a formula $\phi(x, y)$, such that for every $s \subseteq \omega$, there exists $b_s \in \mathbb{M}$ such that $\mathbb{M} \models \phi(a_i, b_s) \iff i \in s$. By Remark I.5.23, P is uniformly stably embedded in $\mathbb{M}$, so there exists a formula $\psi(x, z) \in L^P$ and parameters $c_s \in P$ for every $s \subseteq \omega$, such that $\phi(P, b_s) = \psi(P, c_s)$, and in particular $\mathbb{M} \models \psi(a_i, c_s) \iff i \in s$.

The induced structure on P is inter-definable with the internal L-structure of P (Lemma I.5.21), so there is some formula $\psi'(x, z) \in L$ that defines the same set in P as $\psi(x, z)$, in particular $P \models \psi'(a_i, c_s) \iff i \in s$. The formula $\psi'(x, y)$ has IP in $P \models T$, in contradiction to T being NIP. □

Theorem I.5.34. *If T is NIP, then ACF_T is NIP.*

Proof. Suppose ACF_T has IP, by Fact I.5.31 there is some $\phi(x, y)$ with $|y| = 1$ that has IP modulo ACF_T. Using Fact I.5.30 and compactness, there is an indiscernible sequence $I = (a_i)_{i<|T|^+} \subseteq \mathbb{M}$ and some $c \in \mathbb{M}$ such that $\mathbb{M} \models \phi(a_i, c) \iff i$ is even.

First consider the case where c is transcendental over $P(I)$. One can find, using Ramsey (see e.g. [TZ12, Lemma 5.1.3]), a sequence I' indexed by $|T|^+$ that is indiscernible over c with the same EM-type as I over c — that is, if a formula $\phi(\bar{x}; c)$ holds for every increasing tuple in I, then it holds for every increasing tuple in I' (see [TZ12, Definition 5.1.2]). In particular, c is still transcendental over $P(I')$. Both I and I' are indiscernible and have the same EM-type over the empty set, so there is an automorphism mapping $I' \mapsto I$. If we apply this automorphism on c, then we get c' transcendental over $P(I)$ such that I is indiscernible over c'. By Lemma I.3.14, there is an automorphism fixing $P(I)$ pointwise and mapping $c' \mapsto c$, so I is indiscernible over c, a contradiction.

Now consider the case where c is algebraic over $P(I)$. There is some finite subsequence $I_0 \subseteq I$ and some finite tuple $b \in P$, such that c is algebraic over $I_0 b$. Let $I' \subseteq I$ be some end segment starting after I_0; note that I' is indiscernible over I_0. As P is NIP (Lemma I.5.33), by Fact I.5.32 there is an end segment $I'' \subseteq I'$ that is indiscernible over $I_0 b$. It follows that I'' is also indiscernible over $\mathrm{acl}(I_0 b)$, and in particular over c, a contradiction. □

Example I.5.35. Let ACVF be the theory of algebraically closed valued fields in the divisibility language, that is the language of rings with a binary relation $x|y$ signifying $v(x) < v(y)$. ACVF is NIP, so $\mathrm{ACF}_{\mathrm{ACVF}}$ is NIP.

Remark I.5.36. *One could also use a counting type approach to prove preservation of NIP, similar to the proof of Theorem I.5.24. This would require working* *in a generic extension of ZFC such that* $\mathrm{ded}(\kappa)^{\aleph_0} < 2^\kappa$ *for some infinite cardinal* κ *(where* $\mathrm{ded}(\kappa)$ *is the supremum of cardinalities of linear orders with a dense subset of size* $\leq \kappa$*). For an expanded explanation of this approach, see [She90, Theorem II.4.10] and [Adl07, Corollary 24].*

Alternatively, one could also apply more general results, i.e., [CS15, Corollary 2.5] and [JS20, Proposition 2.5], but we chose to give a direct argument.

I.6 Applications

In this section we will apply the above results to specific theories.

I.6.1 Tuples of algebraically closed fields

In this section we will consider (perhaps infinite) chains of algebraically closed fields, which, for the finite case, is a particular case of *beaux uples* in the sense of [BP88]. The main result of this section is Proposition I.6.4 which classifies the theories of such chains based on the order type of the chain.

Definition I.6.1. For any ordered set I, define $L^I = L_{\text{rings}} \cup \{P_i\}_{i\in I}$ with P_i unitary predicates and define the theory ACF^I expanding ACF in L^I, such that:

1. Each P_i is an algebraically closed field, that is strictly contained in the model.
2. For $i < j$, $P_i \subsetneq P_j$.

In particular, ACF^n is the theory of algebraically closed fields M, with n algebraically closed subfields $P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_{n-1} \subsetneq M$.

Proposition I.6.2. *Let* I *be any ordered set.*

1. *The completions of* ACF^I *are given by fixing the characteristic,* ACF^I_p*.*
2. *Every completion of* ACF^I *is stable.*

Proof. We will first prove for $I = n$, by induction on n. For $n = 0$, $\mathrm{ACF}^0 = \mathrm{ACF}$, and indeed the completions of ACF are given by fixing the characteristic and every completion ACF_p is stable. Suppose it is true for n. We have $\mathrm{ACF}^{n+1} = \mathrm{ACF}_{\mathrm{ACF}^n}$, where we denote the added predicate by P_n. By Proposition I.4.1, the completions of ACF^{n+1} are given by completions of ACF^n, which are given by fixing the characteristic. Furthermore, $\mathrm{ACF}^{n+1}_p = \mathrm{ACF}_{\mathrm{ACF}^n_p}$, so by Theorem I.5.16 every completion ACF^{n+1}_p is stable.

Now consider a general ordered set I and fix a characteristic ACF^I_p. Let ϕ be a sentence in L^I and let $I_\phi \subset I$ be the subset of indexes $i \in I$ such that P_i appears in ϕ. I_ϕ is finite, suppose $I_\phi = \{i_0 < \cdots < i_{n-1}\}$. ACF^n_p is complete, so by renaming the predicates $P_0, \dots, P_{n-1}$ to $P_{i_0}, \dots, P_{i_{n-1}}$ we get that $\mathrm{ACF}^{I_\phi}_p$ is complete. Thus, $\mathrm{ACF}^{I_\phi}_p \vdash \phi$ or $\mathrm{ACF}^{I_\phi}_p \vdash \neg\phi$, but $\mathrm{ACF}^{I_\phi}_p$ is a restriction of ACF^I_p, so $\mathrm{ACF}^I_p \vdash \phi$ or $\mathrm{ACF}^I_p \vdash \neg\phi$. The completions ACF^I_p are all the completions of ACF^I, because any completion has to fix a characteristic so it must extend some ACF^I_p.

We need to show that every completion ACF_p^I is stable. If $\phi \in L^I$ was a formula witnessing instability in ACF_p^I, then it would witness instability in $\mathrm{ACF}_p^{I_\phi}$, which would imply that ACF_p^n is unstable for $n = |I_\phi|$. $\square$

We will further classify the stability of ACF_p^I (when is it ω-stable, superstable or totally transcendental) based on the order type of I. In the case that I is an ordinal, we will need the following lemma.

Lemma I.6.3. *Let α be an ordinal and $M \models \mathrm{ACF}^\alpha$. Any L^β-automorphism of P_β for $\beta < \alpha$ can be extended to an L^α-automorphism of $\mathbb{M}$.*

Proof. Let σ_β be an automorphism of P_β, we will construct by transfinite induction on $\beta \leq \gamma < \alpha$ automorphisms σ_γ of P_γ, such that if $\beta \leq \gamma' < \gamma < \alpha$, then σ_γ extends $\sigma_{\gamma'}$.

Let $\beta \leq \gamma < \alpha$ and suppose we constructed $\sigma_{\gamma'}$ for $\beta \leq \gamma' < \gamma$. Let $\sigma_{<\gamma}$ be the union of $\{\sigma_{\gamma'}\}_{\beta \leq \gamma' < \gamma}$, $\sigma_{<\gamma}$ is a field automorphism of $P_{<\gamma} = \bigcup_{\gamma' < \gamma} P_{\gamma'}$ (if $\gamma = \gamma'+1$ is a successor ordinal, then $\sigma_{<\gamma} = \sigma_{\gamma'}$). Let S be a transcendence basis of P_γ over $P_{<\gamma}$, extend $\sigma_{<\gamma}$ to a field automorphism σ_γ by fixing S pointwise and extending to the algebraic closure. For every $\gamma' < \gamma$, σ_γ preserves $P_{\gamma'}$ setwise, so σ_γ is an L^γ-automorphism.

Once we constructed σ_γ for every $\beta \leq \gamma < \alpha$, we can construct σ_α, an L^α-automorphism of M, in a similar fashion: take $\sigma_{<\alpha}$ the union of $\{\sigma_\gamma\}_{\beta \leq \gamma < \alpha}$, fix a transcendence basis of M over $P_{<\alpha}$ pointwise and extend to the algebraic closure. $\square$

Proposition I.6.4. *For an ordered set I:*

1. *If I is finite, or countable and well-ordered, then every completion of ACF^I is ω-stable.*
2. *If I is uncountable and well-ordered, then every completion of ACF^I is totally transcendental, and in particular superstable, but not ω-stable.*
3. *If I is not well-ordered, then every completion of ACF^I is not superstable.*

Proof. Fix a completion ACF_p^I (by Proposition I.6.2).

1. The theory ACF_p^I depends only on the order type of I, up to renaming predicates, so it is enough to prove for $I = \alpha$ a finite or countable ordinal. We will prove that ACF_p^α is ω-stable by transfinite induction on $\alpha < \omega_1$. For $\alpha = 0$, $\mathrm{ACF}_p^0 = \mathrm{ACF}_p$ is ω-stable. If ACF_p^α is ω-stable, then note that $\mathrm{ACF}_p^{\alpha+1} = \mathrm{ACF}_{\mathrm{ACF}_p^\alpha}$ where we name the added predicate P_α, so by Corollary I.5.26 $\mathrm{ACF}_p^{\alpha+1}$ is ω-stable.

Suppose that α is a countable limit ordinal and for every $\beta < \alpha$, ACF_p^β is ω-stable, the proof that ACF_p^α is ω-stable will be similar to the proof of Theorem I.5.24. Let $\mathbb{M} \models \mathrm{ACF}_p^\alpha$ be a monster model and let $C \subseteq \mathbb{M}$ be a countable subset. Denote $P_{<\alpha} = \bigcup_{\beta<\alpha} P_\beta$. First we will show that every two elements in $\mathbb{M} \setminus \overline{P_{<\alpha}(C)}$ have the same type over C. Let $a_0, a_1 \in \mathbb{M} \setminus \overline{P_{<\alpha}(C)}$, for every $\beta < \alpha$, a_0 and a_1 are transcendental over $P_\beta(C)$ so by Lemma I.3.14 there is an automorphism of $\mathbb{M} \restriction L^{\beta+1}$ preserving $P_\beta(C)$ and mapping $a_0 \mapsto a_1$. Thus, $a_0 \equiv_C^{L^{\beta+1}} a_1$ for every $\beta < \alpha$, so $a_0 \equiv_C^{L^\alpha} a_1$, as every formula in L^α

belongs to some $L^{\beta+1}$ where β is the largest ordinal such that P_β appears in the formula.

Now we will show that there at most countably many types over C realized in $\overline{P_{<\alpha}(C)}$. Any element $a \in \overline{P_{<\alpha}(C)}$ solves some non-zero polynomial of the form $q(x; b, c)$ with $b \in P^n_{<\alpha}$ and $c \in C^m$. There is some $\beta < \alpha$ such that $b \in P^n_\beta$, in particular a satisfies

$$\phi(x; c) = \exists y \in P_\beta \ (q(x; y, c) = 0 \wedge \exists x' q(x'; y, c) \neq 0).$$

Thus, any type in $\overline{P_{<\alpha}(C)}$ contains some formula $\phi(x; c)$ as above. There are countably many formulas in L^α with parameters from C, so it is enough to prove that there are at most countably many types that contain any given formula $\phi(x; c)$ as above.

First of all, P_β is stably embedded in $\mathbb{M}$ (every automorphism of P_β can be extended to an automorphism of $\mathbb{M}$ so we can use Fact I.5.19; alternatively, ACF^α_p is stable so every definable subset is stably embedded), so every C-definable subset of P^n_β is also definable in ACF^α_p with parameters from P_β. Let $D \subseteq P_\beta$ be the set of all the parameters needed to define every C-definable subset of P^n_β. There are at most countably many definable subsets of P^n_β over C, so D is countable.

Let $[\phi] \subseteq S_1^{\mathrm{ACF}^\alpha_p}(C)$ be the set of types implying $\phi(x; c)$ as above, we will construct a map $\rho : [\phi] \to S_n^{\mathrm{ACF}^\beta_p}(D)$ such that ρ has finite fibers. Because ACF^β_p is ω-stable, $|S_n^{\mathrm{ACF}^\beta_p}(D)|$ is countable, so this will imply that $[\phi]$ is countable as needed.

For any type $p(x) \in [\phi]$, choose some realization $a \models p$. In particular, $\models \phi(a; c)$, so we can choose some $b \in P^n_\beta$ such that $q(x; b, c)$ is non-zero and $q(a; b, c) = 0$. Define $\rho(p) = \mathrm{tp}^{\mathrm{ACF}^\beta_p}(b/D)$. Suppose $p_0, p_1 \in [\phi]$ and $\rho(p_0) = \rho(p_1)$, that is, if a_i and b_i are the specific elements we chose for p_i $(i = 0, 1)$, then $b_0 \equiv_D^{\mathrm{ACF}^\beta_p} b_1$. There is an automorphism of P_β over D mapping $b_0 \mapsto b_1$, which can be extended by Lemma I.6.3 to an automorphism of $\mathbb{M}$ over D, so $b_0 \equiv_D^{\mathrm{ACF}^\alpha_p} b_1$. We want to prove that $b_0 \equiv_C^{\mathrm{ACF}^\alpha_p} b_1$. Suppose b_0 belongs to some C-definable set, we can assume that it is a subset of P^n_β because $b_0 \in P^n_\beta$. By the construction of D, this C-definable subset of P^n_β is also D-definable in ACF^α_p, so b_1 belongs to it as $b_0 \equiv_D^{\mathrm{ACF}^\alpha_p} b_1$.

Let $\sigma \in \mathrm{Aut}(\mathbb{M}/C)$ be an automorphism mapping $b_0 \mapsto b_1$. We have $q(\sigma(a_0); b_1, c) = 0$, thus a_0 has the same type over C as a root of $q(x; b_1, c)$, specifically $\sigma(a_0)$. It follows that every type in the fiber of $\rho(p_1)$ is a type over C of a root of $q(x; b_1, c)$, however $q(x; b_1, c)$ is non-zero, so it has only finitely many roots. Thus, ρ has finite fibers.

2. Suppose I is uncountable and well-ordered. If ACF^I_p was not totally transcendental, there would be a binary tree of consistent formulas $\{\phi_s(x; c_s)\}_{s \in 2^{<\omega}}$ (see [TZ12, Definition 5.2.5]). Let $I_0 \subseteq I$ be the finite or countable subset of indexes $i \in I$ such that P_i appears in some formula ϕ_s. The tree $\{\phi_s(x; c_s)\}_{s \in 2^{<\omega}}$ is also a binary tree of consistent formulas in $\mathrm{ACF}^{I_0}_p$, so $\mathrm{ACF}^{I_0}_p$ is not totally transcendental. However, a subset of a well-ordered set is also well-ordered, so by the previous part $\mathrm{ACF}^{I_0}_p$ is ω-stable and in particular totally transcendental.

However, ACF^I can not be ω-stable, as it is not inter-definable with a theory in a countable language — each P_i for $i \in I$ is a distinct definable set.

3. Note that an ordered set I is well-ordered iff I does not contain an infinite descending chain. If I is not well-ordered, let $(i_k)_{k<\omega} \subseteq I$ be a descending chain, then $(P_{i_k})_{k<\omega}$ is a descending chain of definable subfields in ACF_p^I. Considering only the additive group structure, $(P_{i_k})_{k<\omega}$ is a descending chain of definable subgroups each of infinite index in the previous one, so ACF_p^I is not superstable (see e.g. [TZ12, Exercise 8.6.10]). □

I.6.2 Complete system of a Galois group

For a profinite group G one can associate a structure $S(G)$, called the complete system of G, in a multi-sorted language. This definition is due to [CvdDM81], we will present the definition as given in [Ram18, Definition 7.1.6].

Definition I.6.5. Suppose G is a profinite group. Let $\mathcal{N}(G)$ be the collection of open normal subgroups of G. Define

$$S(G) = \coprod_{N \in \mathcal{N}(G)} G/N.$$

Let L_G be the language with a sort X_n for each $n < \omega$, two binary relation symbols $\leq$, C and a ternary relation P. We regard $S(G)$ as an L_G-structure in the following way:

- The coset gN is in the sort X_n iff $[G : N] \leq n$.
- $gN \leq hM$ iff $N \subseteq M$.
- $C(gN, hM)$ iff $N \subseteq M$ and $gM = hM$.
- $P(g_1N_1, g_2N_2, g_3N_3)$ iff $N_1 = N_2 = N_3$ and $g_1g_2N_1 = g_3N_1$.

Note that we do not require the sorts to be disjoint (see [Cha98, §1] for a discussion on the syntax of this structure).

For a field F, let $G(F) = \mathrm{Gal}(\overline{F}/F)$ be the absolute Galois group of F, which is profinite. In [Ram18, Corollary 7.2.7], Ramsey proved that if F is a PAC field such that $\mathrm{Th}(S(G(F))$ is NSOP_1, then $\mathrm{Th}(F)$ is NSOP_1. We will prove the other direction, using the following fact, proved in [Cha02, Proposition 5.5].

Fact I.6.6. *$S(G(F))$ is interpretable in (K, F) where K is any algebraically closed field extending F.*

Proposition I.6.7. *Let F be a PAC field. Then $\mathrm{Th}(F)$ is NSOP_1 iff $\mathrm{Th}(S(G(F)))$ is NSOP_1.*

Proof. The left to right direction is [Ram18, Corollary 7.2.7]

For the right to left direction, let $K \supseteq F$ be a large enough algebraically closed extension, $(K, F) \models \mathrm{ACF}_{\mathrm{Th}(F)}$. From Theorem I.5.9 $\mathrm{ACF}_{\mathrm{Th}(F)}$ is NSOP_1, but from Fact I.6.6 $S(G(F))$ is interpretable in (K, F), so $\mathrm{Th}(S(G(F))$ is NSOP_1. □

I.6.3 Pseudo finite fields

Pseudo finite fields were first studied in [Ax68], we will give the definition from [TZ12].

Definition I.6.8. Suppose F is a field. We say that F is *pseudo-algebraically closed* if every absolutely irreducible variety over F has an F-rational point, or equivalently if it is existentially closed in every regular extension. We say that F is *pseudo-finite* if it is perfect, pseudo-algebraically closed and 1-free (has exactly one extension of degree n for every n). Being pseudo-algebraically closed or pseudo-finite is an elementary property [TZ12, Corollary B.4.3, Remark B.4.12], so there are first-order theories PAC, PSF of pseudo-algebraically closed, pseudo-finite fields respectively.

Proposition I.6.9. $\mathrm{ACF}^{ld}_{\mathrm{PSF}}$ *is model complete.*

Proof. If Q and R are pseudo-finite fields such that $Q \subseteq R$ is a relatively algebraically closed extension, that is $\overline{Q} \cap R = Q$, then $Q \prec R$ [FJ08, Proposition 20.10.2]. In particular, if $Q \subseteq R$ is a regular extension, then it is relatively algebraically closed, so $Q \prec R$. Thus, by Theorem I.4.12, $\mathrm{ACF}^{ld}_{\mathrm{PSF}}$ is model complete. □

Proposition I.6.10. *Every completion of* $\mathrm{ACF}_{\mathrm{PSF}}$ *is simple.*

Proof. By Proposition I.4.1, completions of $\mathrm{ACF}_{\mathrm{PSF}}$ are given by completions of PSF, which are simple by [TZ12, Corollary 7.5.6], so the result follows from Theorem I.5.13. We will give another more direct proof using ACFA, the model companion of difference fields, which is simple [Kim14, Example 2.6.9].

Let $(M, P) \models \mathrm{ACF}_{\mathrm{PSF}}$. We will show that there is an automorphism $\sigma \in \mathrm{Gal}(\overline{P}/P)$ such that $\mathrm{Fix}(\sigma) := \{a \in \overline{P} \mid \sigma(a) = a\} = P$. Consider P_n the unique cyclic extension of degree n of P and σ_n a generator of $\mathrm{Gal}(P_n/P)$. The fixed field of σ_n is P, so the inverse limit of σ_n is an automorphism of $\bar{P}$ whose fixed field is P.

By [Afs14, Corollary 1.2], we can embed $(\overline{P}, \sigma)$ into (N, σ') a model of ACFA, with $\mathrm{Fix}(\sigma') = P$. The structure (N, P) is a reduct of (N, σ'), so it is simple. The structures (M, P), (N, P) and $(\overline{P}, P)$ are models of $\mathrm{ACF}_{\mathrm{PSF}}$, and they can be uniquely expanded to models of $\mathrm{ACF}^{ld}_{\mathrm{PSF}}$. Lemma I.3.10 implies that $(\overline{P}, P) \subseteq (M, P)$, $(\overline{P}, P) \subseteq (N, P)$ are substructures in $\mathrm{ACF}^{ld}_{\mathrm{PSF}}$, because they all share the same predicate. However, Proposition I.6.9 says that $\mathrm{ACF}^{ld}_{\mathrm{PSF}}$ is model complete, so those are elementary substructures. In particular, they are elementary substructures in $\mathrm{ACF}_{\mathrm{PSF}}$. Because (N, P) is simple and $(\bar{P}, P) \prec (N, P)$, we get that $(\bar{P}, P)$ is simple. But also $(\bar{P}, P) \prec (M, P)$, so (M, P) is simple. □

I.7 Questions

There are several questions that arose in our work, which we did not address in this paper.

Question I.7.1. *What other classification properties can we lift from* T *to* ACF_T*?* NTP_2*,* NSOP_n *(for* $n \geq 2$*)?*

Question I.7.2. *What results still hold when we replace* ACF *in* ACF_T *with a different theory of fields?* SCF, ACVF*? The theory of dense pairs of* ACVF *was studied in [Del12].*

Question I.7.3. *What results still hold when we replace* ACF *in* ACF_T *with any strongly minimal theory? See Remark I.5.15.*

Chapter II

Fields with a distinguished submodule

II.1 Introduction

The existentially closed models of a theory are those that are existentially closed in every model extension. Existentially closed models have a random, or generic, aspect to them by their definition — every finite quantifier free structure that exists in some extension will also exist in the existentially closed model. Finding first-order theories that axiomatize the class of existentially closed models is a strong tool in studying the generic models, and if the theory is inductive this will result in the model companion.

In [dE21b], d'Elbée studied the theory of models with a generic substructure. A particular example of interest to us is the theory of fields of positive characteristic with a distinguished sub-vector space over a finite subfield, the class of existentially closed models of this theory is first-order axiomatizable, which gives rise to a model companion. The theory $\mathrm{ACF}_p\mathrm{G}$ of algebraically closed fields of characteristic $p > 0$ with a generic additive subgroup is a specific case of the above construction, as additive subgroups are sub-vector spaces over $\mathbb{F}_p$. This theory was extensively studied in [dE21a].

Furthermore, [dE21b] defines *weak-independence* and *strong-independence*, and gives conditions for a model with a generic substructure to be NSOP_1, where weak-independence is Kim-independence (an introduction to those concepts can be found in the previous chapter). The model companion of fields of positive characteristic with a sub-vector space over a finite subfield satisfies those conditions, so it is NSOP_1. It was also proved that this model companion is not simple ([dE21b, Proposition 5.20]).

It is a natural to try and generalize this results to fields that are of characteristic 0, or vector spaces that are over infinite subfields. Another generalization is to consider modules over infinite subrings (a finite subring is a field). In [dE21b], d'Elbée showed that for fields of characteristic 0 with an additive subgroup (which is a $\mathbb{Z}$-module), the class of existentially closed models is not first-order axiomatizable.

However, it is possible to study the existentially closed models of an inductive theory in a different logical setting, namely in Robinson's logic. In essence, it means that instead of studying models and elementary embeddings between them, we study existentially closed models and embeddings between them (see the introduction of [PY18]). Pillay [Pil00] refers to this setting as the *Category of existentially closed models*. This approach was used by Haykazyan and Kirby [HK21b] in their study of exponential fields — fields F with a distinguished homomorphism $E : F^+ \to F^\times$ from the additive group structure to the multiplicative group structure.

(We note that there is a recent interest in positive model theory, which is a generalization of our setting (e.g. [Hay19, Hru20, DK21]).)

This chapter follows the steps of [HK21b], considering the structure of fields with a submodule. We will first give a description of the existentially closed fields with submodules (Theorem II.3.7). This description will not in general be first-order, except for the case of positive characteristic and submodules over a finite subring (see Remark II.3.8). We will then prove that the category of existentially closed models of this theory is NSOP_1 (see Theorem II.4.8) but not NTP_2 (and in particular, not simple; see Theorem II.4.2); the appropriate definitions for these concepts in the category of existentially closed models appear in Section II.2.5, and are taken from [HK21b]. The proof of NSOP_1 will use weak independence (mentioned above). We will also study strong independence, which does not help with proving NSOP_1 but has interesting properties of its own, including n-amalgamation for every n (see Theorem II.5.5). In the proofs we are using a definition of higher amalgamation that is slightly different from the one found in the literature (see [Hru98, dPKM06, GKK13]). In the appendix we study this notion of amalgamation and its relation to independence.

II.2 Preliminaries

In this section, we will present the definitions and facts needed to work in the category of existentially closed models. Unless stated otherwise, all definition and results will be given as they are presented by Haykazyan and Kirby [HK21b].

II.2.1 Existentially closed models of an inductive theory

Definition II.2.1. A model $M \models T$ is called *existentially closed* if for every extension $M \subseteq N \models T$, and every quantifier-free formula $\phi(x, a)$ with parameters $a \in M$, $N \models \exists x\phi(x, a) \implies M \models \exists x\phi(x, a)$.

Remark II.2.2. *If T is inductive, then for every $A \models T$ we can construct by transfinite induction an extension $A \subseteq M$ such that $M \models T$ is existentially closed.*

Let $\mathrm{Emb}(T)$ be the category of models of T with embeddings between them. Let $\mathcal{EC}(T)$ be the full subcategory of $\mathrm{Emb}(T)$ consisting of existentially closed models and embeddings between them (which in particular preserve existential formulas).

Fact II.2.3 ([HK21b, Fact 2.3])**.** *For two inductive theories T_1 and T_2, the following are equivalent*

1. *The theories T_1 and T_2 have the same universal consequences.*

2. *Every model of T_1 embeds into a model of T_2 and vice-versa.*

3. *The existentially closed models of T_1 and T_2 are the same.*

Two theories T_1 and T_2 satisfying the above equivalent conditions are called *companions*. Thus, $\mathcal{EC}(T)$ uniquely determines the theory T modulo companions for T inductive.

We will also use the following fact.

Fact II.2.4 ([HK21b, Fact 2.2])**.** *Let M be a model of an inductive theory T. The following are equivalent.*

1. *M is an existentially closed model of T.*

2. *For every tuple $a \in M$, the type $\mathrm{tp}_{\exists}^{M}(a)$ is a maximal existential type.*

Remark II.2.5. *In particular, if M is an existentially closed model of T, and $A \subseteq M$ is a subset, then $\mathrm{tp}_{\exists}^{M}(a/A)$ is a maximal existential type over A. Indeed, let M_A be the model M with added constant symbols for A, and let T_A be the same theory as T but in the expanded language. Every model of T_A extending M_A must interpret the constant symbols as A, so M_A is an existentially closed model of T_A, as we allow parameters in the definition of existentially closed. The result then follows from Fact II.2.4.*

II.2.2 Amalgamation and joint embedding

Definition II.2.6. A model $A \models T$ is an *amalgamation base* for $\mathrm{Emb}(T)$ if for every two models $B_1, B_2 \models T$ and embeddings $f_1 : A \to B_1$ and $f_2 : A \to B_2$, then there is a model $C \models T$ and embeddings $g_1 : B_1 \to C$ and $g_2 : B_2 \to C$ such that $g_1 \circ f_1 = g_2 \circ f_2$.

Furthermore, A is a *disjoint amalgamation base* if we can pick g_1, g_2 such that $g_1(B_1) \cap g_2(B_2) = g_1(f_1(A))$.

Fact II.2.7 ([Hod93, Corollary 8.6.8])**.** *Every existentially closed model is a disjoint amalgamation base.*

Definition II.2.8. The category $\mathrm{Emb}(T)$ has the *joint embedding property* (JEP) if any two models of T can be embedded into a third model.

In the category of existentially closed models, extending T to an inductive theory T' with JEP corresponds to choosing a completion in first-order logic. However, we need to make sure that $\mathcal{EC}(T')$ is contained in $\mathcal{EC}(T)$. This gives rise to the following definition.

Definition II.2.9. An inductive extension T' of an inductive theory T is called a *JEP-refinement* of T if T' has JEP and every existentially closed model of T' is an existentially closed model of T

Fact II.2.10 ([HK21b, Lemma 2.12])**.** *If A is an amalgamation base for* $\mathrm{Emb}(T)$*, then $T \cup \mathrm{Th}_{\exists}(A)$ is a JEP-refinement of T.*

Furthermore, an existentially closed model of T is a model of a unique JEP-refinement of T modulo companions.

II.2.3 Higher amalgamation

We proceed to define higher amalgamation, as it was defined in [HK21b].

Let $\mathcal{K} \subseteq \mathrm{Emb}(T)$ be a subcategory. Let $n \geq 3$, consider n as a set $n = \{0, \ldots, n-1\}$ and consider $\mathcal{P}(n)$ and $\mathcal{P}^-(n) = \mathcal{P}(n) \setminus \{n\}$ as categories with a unique morphism $a \to b$ if $a \subseteq b$. Define a $\mathcal{P}(n)$-system (respectively, $\mathcal{P}^-(n)$-system) of $\mathcal{K}$ to be a functor F from $\mathcal{P}(n)$ (respectively, $\mathcal{P}^-(n)$) to $\mathcal{K}$. For each $a \in \mathcal{P}(n)$, denote $F_a = F(a)$.

Suppose that for every $M \in \mathcal{K}$, we have a ternary relation $\mathop{\smile\!\!\!\!\!\mid}$ on subsets of M. A $\mathcal{P}(n)$-system ($\mathcal{P}^-(n)$-system) F is called *independent* with respect to $\mathop{\smile\!\!\!\!\!\mid}$, if for every $a \in \mathcal{P}(n)$ ($a \in \mathcal{P}^-(n)$) and every $b \subseteq a$,

$$F_b \mathop{\smile\!\!\!\!\!\mid}_{\bigcup_{c \subsetneq b} F_c} \bigcup_{b \not\subseteq d \subseteq a} F_d$$

as subsets of F_a, where we consider every embedding $F_b \to F_a$ as an inclusion.

Definition II.2.11. Suppose $\mathcal{K}$, $\mathop{\smile\!\!\!\!\!\mid}$ are as above. Say that $\mathcal{K}$ has *n-amalgamation* ($n \geq 3$) if any independent $\mathcal{P}^-(n)$-system in $\mathcal{K}$ can be completed to an independent $\mathcal{P}(n)$-system. Say that T has n-amalgamation if $\mathrm{Emb}(T)$ has n-amalgamation.

Note that this definition of independent systems and n-amalgamation is not the same as the one used by other authors, e.g. [Hru98, dPKM06, GKK13]. It is, however, similar to the definition of stable systems found in [She90], with the main difference being that in stable systems all embeddings are inclusions and everything lives inside the monster model, so there is no amalgamation. This enables us to use the following fact, which is originally stated for general stable theories, but will be presented here as in [HK21b, Fact 5.3] where it is stated specifically for ACF.

Fact II.2.12 ([She90, Fact XII.2.5])**.** *Let $F = \{F_s\}_{s \subseteq n}$ be an independent $\mathcal{P}(n)$-system of ACF, where every F_s is considered as a subset of F_n, and let $t \subseteq n$. For $i < m$ let $s(i) \in \mathcal{P}(n)$ and let $\bar{a}_i \in F_{s(i)}$. Assume that for some formula $\phi(\bar{x}_0, \ldots, \bar{x}_{m-1})$ we have $F_n \models \phi(\bar{a}_0, \ldots, \bar{a}_{m-1})$. Then there are $\bar{a}'_i \in F_{s(i) \cap t}$ such that $F_n \models \phi(\bar{a}'_0, \ldots, \bar{a}'_{m-1})$, and if $s(i) \subseteq t$, then $\bar{a}'_i = \bar{a}_i$.*

In Appendix A we prove some well known results about higher amalgamation using our definition, including the fact that ACF has n-amalgamation for every n (Proposition A.1.3).

II.2.4 Monster model

We present a notion of saturation for the category of existentially closed models. It is convenient to work inside a large saturated model, which we will call a monster model.

This section borrows from [HK21b, §2.4], except for our definition of strong κ-homogeneity and Proposition II.2.15, see Remark II.2.16.

Definition II.2.13. Let T be an inductive theory with JEP, and suppose M is an existentially closed model of T. Let κ be a cardinal.

- M is called *κ-saturated* if every unitary existential type with parameters from a set $A \subseteq M$ of cardinality less than κ is realized in M.
- M is called *κ-universal* if for every $A \models T$, and a tuple $a \subseteq A$ with $|a| < \kappa$, there exists a tuple $b \subseteq M$ realizing $\mathrm{tp}_\exists^A(a)$ (that is, $\mathrm{tp}_\exists^A(a) \subseteq \mathrm{tp}_\exists^M(b)$).
- M is called *κ-homogeneous* if for any two tuples a, b from M with length less than κ such that $a \equiv^\exists b$, and every singleton $a' \in M$, there exists a singelton $b' \in M$ such that $aa' \equiv^\exists bb'$.
- M is called *strongly κ-homogeneous* if for any two tuples a, b from M with length less than κ such that $a \equiv^\exists b$, there exists an automorphism σ of M such that $\sigma(a) = b$.

Remark II.2.14. *If $\kappa > |L|$, then M is κ-universal iff every model $A \models T$ of size less than κ can be embedded in M, by Löwenheim-Skolem.*

Proposition II.2.15. *In the same settings as above, the following are equivalent:*

1. *M is κ-saturated,*
2. *M is κ^+-universal and κ-homogeneous*
3. *M is $\aleph_0$-universal and κ-homogeneous*

Furthermore, if $\kappa = |M|$, the κ-homogeneity implies strong κ-homogeneity.

Proof. (1) $\Longrightarrow$ (2): Suppose M is κ-saturated. To prove κ^+-universality, let $A \models T$ and let $a = (a_i)_{i<\kappa} \subseteq A$ be a tuple. For $\alpha \leq \kappa$, denote $a_{<\alpha} = (a_i)_{i<\alpha}$. We will construct $b = (b_i)_{i<\kappa}$ satisfying $\mathrm{tp}_\exists^A(a)$, by constructing $b_{<\alpha}$ by induction on $\alpha \leq \kappa$. For $\alpha = 1$, by JEP there is some model $N \models T$ and embeddings $f_1 : A \to N$ and $f_2 : M \to N$. $f_1(a_0)$ realizes $\mathrm{tp}_\exists^A(a_0)$ in N, as it is an existential type. Because M is existentially closed and embeds in N, it follows that $\mathrm{tp}_\exists^A(a_0)$ is consistent in M, and there is $b_0 \in M$ realizing it by saturation. For $\alpha + 1$, consider the existential type $p(x) = \mathrm{tp}_\exists^A(a_\alpha / a_{<\alpha})$, replacing the parameters $a_{<\alpha}$ with $b_{<\alpha}$ results in a consistent existential type $q(x)$ in M, because for every finite conjunction $\psi(x, b_{<\alpha})$ of formulas from $q(x)$, we have $A \models \exists x \psi(x, a_{<\alpha})$, so $M \models \exists x \psi(x, b_{<\alpha})$. By saturation there is some $b_\alpha \in M$ satisfying $q(x)$, so $B_{<\alpha} b_\alpha = b_{<\alpha+1}$ satisfies $\mathrm{tp}_\exists^A(a_{<\alpha+1})$. If α is a limit ordinal, take the union $b_{<\alpha} = \bigcup_{\beta<\alpha} b_{<\beta}$.

To prove κ-homogeneity, suppose $a, b \subseteq M$ are tuples of length less than κ, and let $a' \in M$. Consider $p(x) = \mathrm{tp}_\exists^M(a'/a)$, replacing the parameters a by b results in a consistent existential type, because for every finite conjunction $\psi(x, b)$ of formulas from $q(x)$, we have $A \models \exists x \psi(x, a)$, so $M \models \exists x \psi(x, b)$.

(2) $\Longrightarrow$ (3): trivial.

(3) $\Longrightarrow$ (1): First we will prove that κ-universality and κ-homogeneity imply κ-saturation: Let $p(x)$ be a unitary existential type ($|x| = 1$) over $A \subseteq M$ of size less than κ. There is some extension $N \supseteq M$ with an element $b \in N$ realizing $p(x)$. By κ-universality, there is some $A'b' \subseteq M$ that satisfy $\mathrm{tp}_\exists^N(Ab)$, when considered as tuples. In particular, we have $A' \equiv^\exists A$ in M, so by κ-homogeneity there is some $b'' \in M$ such that $A'b' \equiv^\exists Ab''$. Thus, $b'' \models \mathrm{tp}_\exists^M(b/A) = p(x)$.

Now, assuming $\aleph_0$-universality and κ-homogeneity, will prove by induction on $\lambda \leq \kappa$ that M is λ-saturated. For $\lambda = \aleph_0$, it follows from the above claim.

For λ^+, we know that M is λ-saturated, so by (1) $\implies$ (2) M is λ^+-universal. We also know that M is λ^+-homogeneous, so by the claim M is λ-saturated. For λ a limit cardinal, a set of parameters $A \subseteq M$ of size less than λ, is also of size less than μ for some $\mu < \lambda$.

For the "furthermore" part, if M is $|M|$-homogeneous and $a \equiv^{\exists} b$ in M, we can construct an automorphism mapping a to b by the back and forth method.

□

Remark II.2.16. *Our definition of strong κ-homogeneity differs from the one given in [HK21b], which is*

- *M is called* strongly κ-homogeneous *if for any amalgamation base A of size less than κ and embeddings f_1, f_2 of A in M, there exists an automorphism σ of M such that $\sigma \circ f_1 = f_2$.*

However, strong κ-homogeneity in our definition implies strong κ-homogeneity in their definition: A is an amalgamation base, so there is a model $N \models T$ and embeddings g_1, g_2 of M in N such that $g_1 \circ f_1 = g_2 \circ f_2$. With A considered as a tuple, we have

$$\mathrm{tp}_{\exists}^{M}(f_1(A)) = \mathrm{tp}_{\exists}^{N}(g_1(f_1(A))) = \mathrm{tp}_{\exists}^{N}(g_2(f_2(A))) = \mathrm{tp}_{\exists}^{M}(f_1(A)),$$

because M is existentially closed. From our definition of strong homogeneity, there is an automorphism σ of M such that $\sigma(f_1(A)) = f_2(A)$ considered as tuples, thus $\sigma \circ f_1 = f_2$.

Call M saturated if it is $|M|$-saturated. We will call a large saturated model a *monster model*. In these settings, monster models are often called universal domains, or e-universal domains, but we kept the notation of [HK21b].

We will assume that monster models exist. This usually requires some set theoretic assumptions like the generalized continuum hypothesis, but one can change the set-theoretic universe without changing any object we are interested in, ensuring that monster models of large enough sizes exist. One can also work without a monster model, using only commuting diagrams, but it is less convenient.

II.2.5 Model theoretic tree properties

We will present two properties of formulas, TP_2 and SOP_1, adapted to the category of existentially closed models. The main difference is that the formulas have to be existential, and there must be an existential formula that witnesses inconsistency. In the following, let T be an inductive theory with JEP, and work inside a monster model $\mathbb{M} \models T$.

Definition II.2.17. An existential formula $\phi(x, y)$ (x, y tuples) has TP_2 with respect to $\mathcal{EC}(T)$ if there is an amalgamation base $A \models T$, an existential formula $\psi(y_1, y_2)$ and parameters $(a_{i,j})_{i,j<\omega}$ from A such that the following hold:

1. for all $\sigma \in \omega^\omega$, the set $\left\{\phi(x, a_{i,\sigma(i)}\right\}$ is consistent.

2. $\psi(y_1, y_2)$ implies that $\phi(x, y_1) \wedge \phi(x, y_2)$ is inconsistent, i.e.

$$T \vdash \neg\exists x y_1 y_2 [\psi(y_1, y_2) \wedge \phi(x, y_1) \wedge \phi(x, y_2)]$$

3. for every $i, j, k < \omega$, if $j \neq k$, then $A \models \psi(a_{i,j}, a_{i,k})$.

If no existential formula has TP_2, we say that $\mathcal{EC}(T)$ is NTP_2.

Definition II.2.18. An existential formula $\phi(x, y)$ (x, y tuples) has SOP_1 with respect to $\mathcal{EC}(T)$ if there is an amalgamation base $A \models T$, an existential formula $\psi(y_1, y_2)$ and a binary tree of parameters $(a_\eta)_{\eta \in 2^{<\omega}}$ from A such that the following hold:

1. for every branch $\sigma \in 2^\omega$, the set $\left\{\phi(x, a_{\sigma|_n})\right\}$ is consistent.
2. $\psi(y_1, y_2)$ implies that $\phi(x, y_1) \wedge \phi(x, y_2)$ is inconsistent, i.e.

$$T \vdash \neg \exists x y_1 y_2 [\psi(y_1, y_2) \wedge \phi(x, y_1) \wedge \phi(x, y_2)]$$

3. for all $\eta \in 2^{<\omega}$, if $\nu \trianglerighteq \eta \frown \langle 0 \rangle$, then $A \models \psi(a_{\eta \frown \langle 1 \rangle}, a_\nu)$.

If no existential formula has SOP_1, we say that $\mathcal{EC}(T)$ is NSOP_1.

Remark II.2.19. *If the class $\mathcal{EC}(T)$ is first-order axiomatizable by T', that is T' is the model companion of T, then the above definitions are equivalent to T' being NTP_2, $NSOP_1$ respectively in the usual first-order sense.*

There is a Kim-Pillay style characterization for NSOP_1 theories in the category of existentially closed models. This characterization is due to Haykazyan and Kirby [HK21b], and is based on a theorem of Chernikov and Ramsey [CR16] for complete first-order theories.

Fact II.2.20 ([HK21b, Theorem 6.4])**.** *Let $\mathop{\smile\!\!\!\!\!\mid}$ be a* $\mathrm{Aut}(\mathbb{M})$ *ternary relation on small subsets of* $\mathbb{M}$. *Assume that $\mathop{\smile\!\!\!\!\!\mid}$ satisfies the following, for any small existentially closed model M and tuples a, b from* $\mathbb{M}$*:*

- *(Strong finite character) if $a \not\mathop{\smile\!\!\!\!\!\mid}_M b$, then there is an existential formula* $\phi(x, b, m) \in \mathrm{tp}_\exists(a/Mb)$ *such that for any a' realizing ϕ, the relation $a' \not\mathop{\smile\!\!\!\!\!\mid}_M b$ holds.*
- *(Existence over models) $a \mathop{\smile\!\!\!\!\!\mid}_M M$*
- *(Monotonicity) $aa' \mathop{\smile\!\!\!\!\!\mid}_M bb'$ implies $a \mathop{\smile\!\!\!\!\!\mid}_M b$.*
- *(Symmetry) $a \mathop{\smile\!\!\!\!\!\mid}_M b$ implies $b \mathop{\smile\!\!\!\!\!\mid}_M a$.*
- *(Independence theorem) If $c_1 \mathop{\smile\!\!\!\!\!\mid}_M c_2$, $b_1 \mathop{\smile\!\!\!\!\!\mid}_M c_1$, $b_2 \mathop{\smile\!\!\!\!\!\mid}_M c_2$ and $b_1 \equiv^\exists_M b_2$ then there exists b with $b \equiv^\exists_{Mc_1} b_1$ and $b \equiv^\exists_{Mc_2} b_2$.*

Then $\mathcal{EC}(T)$ has $NSOP_1$.

It is folklore that the independence theorem is equivalent to 3-amalgamation, see e.g. the discussion under [Kim14, Definition 9.1.3]. However, different definitions of amalgamation are used by different authors, as noted in the beginning of Section II.2.3. For this reason we include in the Appendix a proof of this equivalence in our setting (Proposition A.2.1).

II.3 Special models of fields with a submodule

In this section, we will define the theory of fields with a submodule, and give a characterization of special models that we are interested in: existentially closed models and amalgamation bases.

II.3.1 Existentially closed models

For the rest of the paper, let R be an integral domain.

Lemma II.3.1. *If A, B, C are R-modules such that $B \subseteq A$, then $A \cap (B + C) = B + (A \cap C)$.*

Proof. It is clear that $B + (A \cap C) \subseteq A \cap (B + C)$. For the other inclusion, suppose $a \in A \cap (B + C)$. There are $b \in B$ and $c \in C$ such that $a = b + c$, but $b \in b \subseteq A$, so we get $c = a - b \in A \cap C$. Thus, $a \in B + (A \cap C)$. ☐

Definition II.3.2. Let $L_{R;P}$ be the language of rings with a constant symbol for every element $r \in R$, and a unitary predicate P. Define the theory[1] $\mathrm{F}_{R\text{-module}}$ in the language $L_{R;P}$, to be the theory of fields with the quantifier-free diagram of R, and P an R-module.

Remark II.3.3. *If $M, N \models \mathrm{F}_{R\text{-module}}$, then*

1. *R is a subring of M.*
2. *M is an $L_{R;P}$-substructure of N iff M is a subfield of N and $P_N \cap M = P_M$*

Example II.3.4. If $R = \mathbb{Z}$, then $\mathrm{F}_{\mathbb{Z}-\text{module}}$ is the theory of fields of characteristic 0 with an additive subgroup. If $R = \mathbb{Q}$, then $\mathrm{F}_{\mathbb{Q}-\text{module}}$ is the theory of fields of characteristic 0 with a divisible additive subgroup. If $R = \mathbb{F}_p$, then $\mathrm{F}_{\mathbb{F}_p-\text{module}}$ is the theory of fields of characteristic p with an additive subgroup, which was studied in [dE21b].

Definition II.3.5. Let K be a field containing R. Call a variety $V \subseteq K^n$ *R-free* if there is some field extension $K' \supseteq K$, and $a \in K'^n$ a generic point of V over K, such that a is R-linearly independent over K. That is, if $r_0a_0 + \cdots + r_{n-1}a_{n-1} \in K$ for $r_i \in R$, then $r_0 = \cdots = r_{n-1} = 0$.

Definition II.3.6. Let $M \models \mathrm{F}_{R\text{-module}}$ be a field with a sub-R-module, and let $0 \leq k \leq n$, $0 \leq s$. For a matrix $A \in \mathrm{Mat}_{n\times s}(R)$ and a tuple $c \in M^n$, call (A, c) a *k-compatible pair* if for every $r_0, \ldots, r_{k-1} \in R$,

1. $r_0A_0 + \cdots + r_{k-1}A_{k-1} = 0 \implies r_0c_0 + \cdots + r_{k-1}c_{k-1} \in P_M$,
2. for $k \leq i < n$, either $A_i \neq r_0A_0 + \cdots + r_{k-1}A_{k-1}$, or $r_0c_0 + \cdots + r_{k-1}c_{k-1} - c_i \notin P_M$,

where A_i is the i-th row of A.

Theorem II.3.7. *Let $M \models \mathrm{F}_{R\text{-module}}$ be a field with a sub-R-module. The model M is existentially closed iff for every R-free variety $V \subseteq M^s$ and k-compatible pair (A, c), where $A \in \mathrm{Mat}_{n\times s}(R)$, $c \in M^n$, there is a point $b \in V$ such that for $a = Ab + c$, $a_0, .., a_{k-1} \in P_M$ and $a_k, .., a_{n-1} \notin P_M$.*

[1] F stands for the theory of fields, as ACF stands for the theory of algebraically closed fields.

Proof. For the left to right implication, suppose V, A, c are as above. There is some field extension $M' \supseteq M$ with $b' \in M'$, such that b' is a generic point of V over M. Let $a' = Ab' + c$, and consider M' as a model of $\mathrm{F}_{R\text{-module}}$, with $P_{M'} = P_M + \langle a'_0, \ldots, a'_{k-1}\rangle_R$. To show that M is an $L_{R,P}$-substructure of M', we need to show that $P_{M'} \cap M = P_M$. By Lemma II.3.1, $P_{M'} \cap M = (P_M + \langle a'_0, \ldots, a'_{k-1}\rangle_R) \cap M = P_M + (\langle a'_0, \ldots, a'_{k-1}\rangle_R \cap M)$, so it is enough to show $\langle a'_0, \ldots, a'_{k-1}\rangle_R \cap M \subseteq P_M$. Suppose $m \in \langle a'_0, \ldots, a'_{k-1}\rangle_R \cap M$, we can write $m = r_0 a'_0 + \cdots + r_{k-1}a'_{k-1}$ with $r_i \in R$. Substitute a'_i for $A_i b' + c_i$ and rearrange to get

$$(r_0 A_0 + \cdots + r_{k-1}A_{k-1})b' = m - (r_0 c_0 + \cdots + r_{k-1}c_{k-1}) \in M.$$

However, b' is R-linearly independent over M, so we must have $r_0 A_0 + \cdots + r_{k-1}A_{k-1} = 0$. This implies that $m = r_0 c_0 + \cdots + r_{k-1}c_{k-1}$, and by k-compatibility $r_0 c_0 + \cdots + r_{k-1}c_{k-1} \in P_M$, so $m \in P_M$.

Consider the formula

$$\phi(y) = V(y) \wedge \bigwedge_{i<k} A_i y + c_i \in P \wedge \bigwedge_{k \le i < n} A_i y + c_i \notin P$$

where $|y| = |b'|$. We want to show that $\phi(y)$ is satisfied by b' in M'. It is obvious that $b' \in V(M')$ and $A_i b' + c_i = a'_i \in P_{M'}$ for $i < k$, it remains to prove that $A_i b' + c_i = a'_i \notin P_{M'}$ for $k \le i < n$. Assume towards contradiction that $a'_i \in P_{M'}$ for some $k \le i < n$, then there are $r_0, \ldots, r_{k-1} \in R$ and $p \in P_M$ such that $a'_i = p + r_0 a'_0 + \cdots + r_{k-1}a'_{k-1}$. Substitute a'_j for $A_j b' + c_j$ and rearrange to get

$$(A_i - r_0 A_0 + \cdots - r_{k-1}A_{k-1})b' = p + r_0 c_0 + \cdots + r_{k-1}c_{k-1} - c_i \in M.$$

Again, because b' is R-linearly independent over M, this implies $A_i - r_0 A_0 + \cdots - r_{k-1}A_{k-1} = 0$, so $A_i = r_0 A_0 + \cdots + r_{k-1}A_{k-1}$. It also follows that $p + r_0 c_0 + \cdots + r_{k-1}c_{k-1} - c_i = 0$, so $r_0 c_0 + \cdots + r_{k-1}c_{k-1} - c_i = -p \in P_M$, in contradiction to k-compatibility. $\phi(y)$ is satisfied by b' in M', so by existential closeness there is some $b \in V(M)$, such that $a = Ab + c$ satisfies $a_0, .., a_{k-1} \in P_M$, $a_k, .., a_{n-1} \notin P_M$, as needed.

For the right to left implication, let $M \models \mathrm{F}_{R\text{-module}}$ satisfy the right-hand condition, and let $M' \models \mathrm{F}_{R\text{-module}}$ be some model extending M. We need to show that for every formula $\psi(x)$ which is a conjunction of atomic formulas, where $x = (x_0, \ldots, x_{n-1})$ is a tuple of variables, if $M' \models \exists x \psi(x)$, then $M \models \exists x \psi(x)$. Atomic formulas in $L_{R;P}$ take one of the following forms:

1. $q(x) = 0$,
2. $q(x) \neq 0$,
3. $q(x) \in P$,
4. $q(x) \notin P$,

where $q(x)$ is a polynomial over M. Let $\psi(x)$ be a conjunction of atomic formulas. By introducing more variables, we can replace the atomic formulas of the second form $q(x) \neq 0$ with $x_n \cdot q(x) = 1$, to get an atomic formula of the first form, because $\exists x(q(x) \neq 0) \iff \exists x, x_n(x_n \cdot q(x) = 1)$. Similarly we can replace the third and fourth forms $q(x) \in P$, $q(x) \notin P$ with $q(x) = x_n \wedge x_n \in P$, $q(x) = x_n \wedge x_n \notin P$. After those replacements, we are left only with atomic formulas of the forms $q(x) = 0$, $x_i \in P$, and $x_i \notin P$.

Furthermore, suppose $a' \in M'$ witnesses the existence $M' \models \exists x\psi(x)$. For every $i < n$, either $a'_i \in P$ or $a'_i \notin P$. Taking the conjunction of ψ with the corresponding conditions $x_i \in P$ or $x_i \notin P$, we get a stronger formula $\psi^*(x)$ that is still satisfied by a', and has the additional property that for every $i < n$ either $x_i \in P$ or $x_i \notin P$ appears in $\psi^*(x)$. Thus, it is enough to prove existential closeness for formulas with the above property, and we can assume that $\psi(x)$ is of the form

$$\psi(x) = W(x) \wedge x_0, .., x_{k-1} \in P \wedge x_k, .., x_{n-1} \notin P$$

where $W(x)$ is a conjunction of polynomial equations, i.e. a variety over M.

Let $a' \in M'$ witness the existence $M' \models \exists x\psi(x)$. Consider the fraction field of R, $\mathrm{Frac}(R) \subseteq M$. There is some $0 \leq s \leq n$, and some tuple $b' \in M'^s$ that is $\mathrm{Frac}(R)$-linearly independent over M, such that $M + \langle a' \rangle_{\mathrm{Frac}(R)} = M \oplus \langle b' \rangle_{\mathrm{Frac}(R)}$. Write $a' = Ab' + c$ for $A \in \mathrm{Mat}_{n\times s}(\mathrm{Frac}(R))$, $c \in M^n$. We can assume without loss of generality that A is a matrix over R, else let $0 \neq d \in R$ be the product of the denominators of all elements in A, and replace A, b' with $dA, \frac{1}{d}b'$ to get a matrix over R.

We will prove that (A, c) is a k-compatible pair. Let $r_1, \dots, r_k \in R$, if $r_1A_1 + \cdots + r_kA_k = 0$, then

$$\begin{aligned} P_{M'} \ni r_1a'_1 + \cdots + r_ka'_k &= (r_1A_1 + \cdots + r_kA_k)b' + r_1c_1 + \cdots + r_kc_k \\ &= r_1c_1 + \cdots + r_kc_k \end{aligned}$$

but also $r_1c_1 + \cdots + r_kc_k \in M$, so $r_1c_1 + \cdots + r_kc_k \in P_M$. Suppose for $k \leq i < n$ that both $A_i = r_1A_1 + \cdots + r_kA_k$ and $r_1c_1 + \cdots + r_kc_k - c_i \in P_M$, then

$$\begin{aligned} r_1a'_1 + \cdots + r_ka'_k - a'_i &= (r_1A_1 + \cdots + r_kA_k - A_i)b' + r_1c_1 + \cdots + r_kc_k - c_i \\ &= r_1c_1 + \cdots + r_kc_k - c_i \in P_M. \end{aligned}$$

Thus, $a'_i \in P_M + \langle a'_1, .., a'_k \rangle_R \subseteq P_{M'}$, a contradiction. Let V be the locus of b' over M. The pair (A, c) is k-compatible and V is R-free, so by our assumption there exists $b \in V$, such that for $a = Ab + c$ we have $a_1, .., a_k \in P_M$, $a_{k+1}, .., a_n \notin P_M$. Furthermore, $W(Ay + c)$ is contained in $V(y)$, as $a' = Ab' + c$ belongs to W, so $a = Ab + c \in W$. We found $a \in M^n$ such that $M \models \psi(a)$, as needed. ☐

Remark II.3.8. *Let M be an existentially closed model of $F_{R-module}$. Then R is definable as a subset of M. Indeed, $R = \{x \in M \mid xP_M \subseteq P_M\}$: '$\subseteq$' is clear. For the other direction assume by contradiction that $m \in M \setminus R$ and $mP_M \subseteq P_M$, and consider the structure $N = (\overline{M(t)}, P_M + R\frac{t}{m})$, where t is transcendental over M. Then it is easy to check that N is an extension of M and that $t \notin P_N$. Thus, $N \models \exists x(x \in P \wedge mx \notin P)$ (the second conjunct uses the assumption towards contradiction), and since M is existentially closed, we get a contradiction. This is essentially the same proof as [dE21b, Proposition 5.32].*

It follows that if R is infinite, then the class of existentially closed models of $F_{R-module}$ is not elementary — else, starting with some existentially closed model, we could construct by compactness an existentially closed model extending *it with a strictly larger R. On the other hand, if R is finite, then this class is elementary, by [dE21b, Proposition 5.4]. In particular, if R is infinite, the characterization of existentially closed models given in Theorem II.3.7 is not first-order.*

II.3.2 Amalgamation bases

Theorem II.3.9. *The amalgamation bases of $F_{R\text{-module}}$ are precisely the algebraically closed fields with sub-R-modules, $\mathrm{ACF}_{R\text{-module}}$. Furthermore, they are disjoint amalgamation bases.*

Proof. Let M be an algebraically closed fields with a sub-R-module, and $f_1 : M \to M_1$, $f_2 : M \to M_2$ be embeddings of fields with sub-R-modules. There is an algebraically closed field N and field embeddings $g_1 : M_1 \to N$, $g_2 : M_2 \to N$ such that $g_1 \circ f_1 = g_2 \circ f_2$ and $M_1 \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{ACF}}_{M} M_2$ in N, where we identify the fields with their images under the embeddings. In particular, $M_1 \cap M_2 = M$, because M is algebraically closed.

Give N an $L_{R;P}$ structure by defining $P_N = P_{M_1} + P_{M_2}$. To show that M_1, M_2 are $L_{R;P}$ substructures of N, we need to show that $M_1 \cap P_N = P_{M_1}$, and $M_2 \cap P_N = P_{M_2}$ will follow from symmetry. By Lemma II.3.1, $M_1 \cap P_N = M_1 \cap (P_{M_1} + P_{M_2}) = P_{M_1} + (M_1 \cap P_{M_2})$, and we have $M_1 \cap P_{M_2} = M_1 \cap M_2 \cap P_{M_2} = M \cap P_{M_2} = P_M$, so $M_1 \cap P_N = P_{M_1} + P_M = P_{M_1}$. Thus, M is an amalgamation base, and from $M_1 \cap M_2 = M$ it is a disjoint amalgamation base.

Suppose $M \models \mathrm{F}_{R\text{-module}}$ is not algebraically closed. There is an element $a \in \overline{M} \setminus M$. Let $M_1 = M_2 = \overline{M}$, but define $P_{M_1} = P_M$, $P_{M_2} = P_M + \langle a \rangle_R$. Note that $P_{M_2} \cap M = P_M$, because if $p + ra \in (P_M + \langle a \rangle_R) \cap M$, then $ra \in M$, so $r = 0$, which implies $p + ra = p \in P_M$; thus $M \subseteq M_2$ is an $L_{R;P}$-substructure. Suppose we could amalgamate M_1, M_2 to a model $N \models \mathrm{F}_{R\text{-module}}$ by embeddings $g_1 : M_1 \to N$, $g_2 : M_2 \to N$, such that $g_1|_M = g_2|_M$. By changing N, we can assume that g_2 is an inclusion $M_2 \subseteq N$, and in particular $a \in P_{M_2} \subseteq P_N$. However, $M_1 = \overline{M}$, so $\mathrm{Im}(g_1) = \overline{M}$ and there is some $b \in M_1$ such that $g_1(b) = a$. In particular, $b \in P_{M_1} = P_M$. This would imply $a = g_1(b) = b \in M$, as $g_1|_M = \mathrm{id}_M$, a contradiction. Thus, M is not an amalgamation base. □

II.4 Classification

II.4.1 TP_2

We will construct a formula that is TP_2 in every JEP refinement of $\mathrm{F}_{R\text{-module}}$, as per Definition II.2.17. In particular, this will prove that for every JEP refinement T, $\mathcal{EC}(T)$ is *not* NTP_2.

Lemma II.4.1. *If $M \models \mathrm{F}_{R\text{-module}}$ is existentially closed, then the index $[M : P_M] = \infty$.*

Proof. If $M' \supseteq M$ is a large enough field extension, then in particular $[M' : P_M] = \infty$. Consider the $L_{R;P}$-structure on M' given by $P_{M'} = P_M$, M is an $L_{R;P}$-substructure of M'. The fact that $[M' : P_{M'}] = \infty$ can be expressed by existential sentences "there exist at least n elements in different P-cosets" for every n, so by existential closeness we have $[M : P_M] = \infty$. □

Theorem II.4.2. *Let T be some JEP refinement of $F_{R\text{-module}}$. The formula $\phi(x; y, z) = y \cdot x + z \in P$ has TP_2 with respect to $\mathcal{EC}(T)$.*

Proof. Take the formula $\psi(y_1, z_1; y_2, z_2)$ to be $y_1 = y_2 \wedge z_1 - z_2 \notin P$. Let $M \models T$ be existentially closed such that $|M| > |R| + \aleph_0$, in particular it is an amalgamation base, and it is existentially closed in $\mathrm{F}_{R\text{-module}}$. The fact that $|M| > |R| + \aleph_0$ implies in particular that $\dim_{\mathrm{Frac}(R)}(M) \geq \aleph_0$, so there are $\beta_1, \beta_2, \dots \in M$ such that $1, \beta_1, \beta_2, \dots$ are Frac(R)-linearly independent, and in particular R-linearly independent. By Lemma II.4.1, $[M : P_M] = \infty$, so there are $\gamma_1, \gamma_2, \dots \in M$ that are all in different P_M-cosets. Take the sequence of tuples $a_{ij} = (\beta_i, \gamma_j)$. Conditions (2) and (3) of Definition II.2.17 obviously hold, it remains to show (1). Let $\sigma \in (\omega \setminus \{0\})^{\omega \setminus \{0\}}$, by compactness it is enough to show that for every $n > 0$, $\bigwedge_{i=1}^{n} (\beta_i \cdot x + \gamma_{\sigma(i)} \in P)$ is consistent.

Consider the variety $V(x_0, x_1, \dots, x_n)$ given by

$$x_1 = \beta_1 x_0,$$
$$\vdots$$
$$x_n = \beta_n x_0.$$

Let $b' = (b'_0, \beta_1 b'_0, \dots, \beta_n b'_0)$ be a generic point of V in some field extension. Suppose that for $r_0, \dots, r_n \in R$ we have

$$r_0 b'_0 + r_1 \beta_1 b'_0 + \dots + r_n \beta_n b'_0 \in M,$$

$$(r_0 + r_1 \beta_1 + \dots + r_n \beta_n) b'_0 \in M,$$

then $r_0 + r_1\beta_1 + \dots + r_n\beta_n = 0$, else we would get $b'_0 \in M$. But $1, \beta_1, \dots, \beta_n$ are R-linearly independent, so we must have $r_0 = \dots = r_n = 0$, thus V is R-free. Consider the $n \times (n+1)$ matrix

$$A = \begin{pmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \end{pmatrix}$$

and the tuple $c = (\gamma_{\sigma(1)}, \dots, \gamma_{\sigma(n)})$. We claim that A and c are n-compatible. It is enough to see that the matrix A is of rank n, so if $r_1 A_1 + \dots + r_n A_n = 0$, then $r_1 = \dots = r_n = 0$. From Theorem II.3.7, it follows that there is a point $b = (b_0, \beta_1 b_0, \dots, \beta_n b_0) \in V$ such that for

$$Ab + c = (\beta_1 b_0 + \gamma_{\sigma(1)}, \dots, \beta_n b_0 + \gamma_{\sigma(n)})$$

we have $\beta_i b_0 + \gamma_{\sigma(i)} \in P_M$, as needed. ☐

Remark II.4.3. *The above implies in particular that every JEP refinement of $F_{R\text{-module}}$ is non-simple, by [HK21b, Proposition A.5], where the definition for simplicity in the category of existentially closed models is given in [HK21b, Definition A.4].*

II.4.2 NSOP$_1$

We will show that every for every JEP refinement T of $\mathrm{F}_{R\text{-module}}$, $\mathcal{EC}(T)$ is NSOP$_1$, using Fact II.2.20. The independence relation that we will use was defined in [dE21b, Definition 3.1], and called weak independence. We will present the definition for our specific case.

Definition II.4.4. For a model $M \models \mathrm{ACF}_{R\text{-module}}$, and subsets $A, B, C \subseteq M$, say that A and B are *weakly independent* over C, and denote $A \mathop{\smile\!\!\!\!\!\mid}\,^{w}_{C} B$, if $A \mathop{\smile\!\!\!\!\!\mid}\,^{\mathrm{ACF}}_{C} B$ and $P_M \cap (\overline{AC} + \overline{BC}) = P_M \cap \overline{AC} + P_M \cap \overline{BC}$.

Remark II.4.5. *The inclusion $P_M \cap \overline{AC} + P_M \cap \overline{BC} \subseteq P_M \cap (\overline{AC} + \overline{BC})$ is always true.*

Lemma II.4.6 (3-amalgamation)**.** $\mathrm{ACF}_{R\text{-module}}$ *has* 3*-amalgamtion, meaning any weakly independent $\mathcal{P}^-(3)$-system of* $\mathrm{ACF}_{R\text{-module}}$ *can be completed to a weakly independent $\mathcal{P}(3)$-system.*

Proof. Suppose $M = \{M_s\}_{s \in \mathcal{P}^-(3)}$ is a weakly independent $\mathcal{P}^-(3)$-system of $\mathrm{ACF}_{R\text{-module}}$, and denote $P_s = P_{M_s}$. By Proposition A.1.3 there is some algebraically closed field M_3 that completes M as an independent system of ACF. By embedding all the system in M_3, we can assume that the embeddings are inclusions. Define $P_3 = P_{\hat{0}} + P_{\hat{1}} + P_{\hat{2}}$, where $\hat{i} = 3 \setminus \{i\}$, and consider (M_3, P_3) as a model of $\mathrm{ACF}_{R\text{-module}}$. We need to show that M_3 is an $L_{R;P}$-extension of the rest of the system, that is that $P_3 \cap M_{\hat{i}} = P_{\hat{i}}$. By symmetry it is enough to prove for $i = 0$.

By Lemma II.3.1, $P_3 \cap M_{\hat{0}} = (P_{\hat{0}} + P_{\hat{1}} + P_{\hat{2}}) \cap M_{\hat{0}} = P_{\hat{0}} + (P_{\hat{1}} + P_{\hat{2}}) \cap M_{\hat{0}}$, so it is enough to prove $(P_{\hat{1}} + P_{\hat{2}}) \cap M_{\hat{0}} \subseteq P_{\hat{0}}$. Let $m_{\hat{0}} \in (P_{\hat{1}} + P_{\hat{2}}) \cap M_{\hat{0}}$, there are $p_{\hat{1}} \in P_{\hat{1}}$, $p_{\hat{2}} \in P_{\hat{2}}$ such that $m_{\hat{0}} = p_{\hat{1}} + p_{\hat{2}}$. Fact II.2.12 applied for $t = \hat{1}$ implies that there exists $m_{\{0\}} \in M_{\{0\}}$, $m_{\{2\}} \in M_{\{2\}}$ such that $m_{\{2\}} = p_{\hat{1}} + m_{\{0\}}$, in particular $p_{\hat{1}} \in M_{\{0\}} + M_{\{2\}}$. However, by weak independence in $M_{\hat{1}}$, $P_{\hat{1}} \cap (M_{\{0\}} + M_{\{2\}}) = P_{\{0\}} + P_{\{2\}}$, so $p_{\hat{1}} \in P_{\{0\}} + P_{\{2\}}$. Similarly, by applying Fact II.2.12 for $t = \hat{2}$, we get $p_{\hat{2}} \in P_{\{0\}} + P_{\{1\}}$. Altogether, $m_{\hat{0}} = p_{\hat{1}} + p_{\hat{2}} \in P_{\{0\}} + P_{\{1\}} + P_{\{2\}}$. By Lemma II.3.1, $(P_{\{0\}} + P_{\{1\}} + P_{\{2\}}) \cap M_{\hat{0}} = P_{\{0\}} \cap M_{\hat{0}} + (P_{\{1\}} + P_{\{2\}})$, but $P_{\{1\}} + P_{\{2\}} \subseteq P_{\hat{0}}$, so it is enough to prove $P_{\{0\}} \cap M_{\hat{0}} \subseteq P_{\hat{0}}$. ACF-independence of the $\mathcal{P}(3)$-system implies that $M_{\{0\}} \mathop{\smile\!\!\!\!\!\mid}\,^{\mathrm{ACF}}_{M_\emptyset} M_{\hat{0}}$ in M_3, and $M_\emptyset$ is algebraically closed so $M_{\{0\}} \cap M_{\hat{0}} = M_\emptyset$. Thus,

$$P_{\{0\}} \cap M_{\hat{0}} = P_{\{0\}} \cap M_{\{0\}} \cap M_{\hat{0}} = P_{\{0\}} \cap M_\emptyset = P_\emptyset \subseteq P_{\hat{0}}.$$

It remains to show that the system is weakly independent. By symmetry, there are only two general cases we need to check

1. $M_{\hat{0}} \mathop{\smile\!\!\!\!\!\mid}\,^{w}_{M_{\{1\}}M_{\{2\}}} M_{\hat{1}} M_{\hat{2}}$,

2. $M_{\hat{0}} \mathop{\smile\!\!\!\!\!\mid}\,^{w}_{M_\emptyset} M_{\{0\}}$.

Because the system is ACF-independent, we already have $M_{\hat{0}} \mathop{\smile\!\!\!\!\!\mid}\,^{\mathrm{ACF}}_{M_{\{1\}}M_{\{2\}}} M_{\hat{1}} M_{\hat{2}}$, $M_{\hat{0}} \mathop{\smile\!\!\!\!\!\mid}\,^{\mathrm{ACF}}_{M_\emptyset} M_{\{0\}}$. For the first case, notice that $P_3 \cap M_{\hat{0}} = P_{\hat{0}}$, $P_3 \cap \overline{M_{\hat{1}} M_{\hat{2}}} \supseteq P_{\hat{1}} + P_{\hat{2}}$, so

$$P_3 \cap M_{\hat{0}} + P_3 \cap \overline{M_{\hat{1}} M_{\hat{2}}} \supseteq P_{\hat{0}} + P_{\hat{1}} + P_{\hat{2}} = P_3 \supseteq P_3 \cap (M_{\hat{0}} + \overline{M_{\hat{0}} M_{\hat{1}}})$$

where the other inclusion is obvious (Remark II.4.5). For the second case, by Lemma II.3.1 $P_3 \cap (M_{\hat{0}} + M_{\{0\}}) = (P_{\hat{0}} + P_{\hat{1}} + P_{\hat{2}}) \cap (M_{\hat{0}} + M_{\{0\}}) = P_{\hat{0}} + (P_{\hat{1}} + P_{\hat{2}}) \cap (M_{\hat{0}} + M_{\{0\}})$, so it is enough to show $(P_{\hat{1}} + P_{\hat{2}}) \cap (M_{\hat{0}} + M_{\{0\}}) \subseteq P_{\hat{0}} + P_{\{0\}}$. Let $p_{\hat{1}} + p_{\hat{2}} \in (P_{\hat{1}} + P_{\hat{2}}) \cap (M_{\hat{0}} + M_{\{0\}})$, where $p_{\hat{1}} \in P_{\hat{1}}$, $p_{\hat{2}} \in P_{\hat{2}}$. We can write $p_{\hat{1}} + p_{\hat{2}} = m_{\hat{0}} + m_{\{0\}}$, where $m_{\hat{0}} \in M_{\hat{0}}$, $m_{\{0\}} \in M_{\{0\}}$. By Fact II.2.12 applied for $t = \hat{1}$, there are $m'_{\{0\}} \in M_{\{0\}}$, $m_{\{2\}} \in M_{\{2\}}$ such that $p_{\hat{1}} + m'_{\{0\}} = m_{\{2\}} + m_{\{0\}}$, so $p_{\hat{1}} \in M_{\{0\}} + M_{\{2\}}$. By weak independence in $M_{\hat{1}}$, $P_{\hat{1}} \cap (M_{\{0\}} + M_{\{2\}}) = P_{\{0\}} + P_{\{2\}}$, so $p_{\hat{1}} \in P_{\{0\}} + P_{\{2\}}$. Similarly, by applying Fact II.2.12 for $t = \hat{2}$, $p_{\hat{2}} \in P_{\{0\}} + P_{\{1\}}$. Thus, $p_{\hat{0}} + p_{\hat{2}} \in P_{\{0\}} + P_{\{1\}} + P_{\{2\}} \subseteq P_{\hat{0}} + P_{\{0\}}$. □

Lemma II.4.7. *Suppose $\mathbb{M}$ is a monster model of a JEP-refinement of $F_{R\text{-module}}$. For a singleton $a \in \mathbb{M}$, a tuple $b \in \mathbb{M}$ and $C \subseteq \mathbb{M}$, if $a \in \overline{C(b)}$, then there is a formula $\phi(x, b, c) \in tp_\exists(a/Cb)$ isolating the type, in the sense that if $a' \models \phi(x, b, c)$ then $a' \equiv^\exists_{Cb} a$.*

Furthermore, we can choose $\phi(x, b, c)$ in such a way that for any $a', b' \in \mathbb{M}$, $\models \phi(a', b', c)$ implies $a' \in \overline{C(b')}$.

Proof. We have $a \in \overline{C(b)}$, so there is some non-zero polynomial $q(x, b, c)$ with a as a root, where $c \in C$. In particular, the formula $q(x, b, c) = 0$ belongs to $\mathrm{tp}_\exists(a/Cb)$ and has finitely many realizations. Take some formula $\phi(x, b, c) \in \mathrm{tp}_\exists(a/Cb)$ with a minimal number of realizations, a conjunction of existential formulas is existential, so $\phi(x, b, c)$ must imply every formula in $\mathrm{tp}_\exists(a/Cb)$. Let $a' \models \phi(x, b, c)$, it follows that $a' \models \mathrm{tp}_\exists(a/Cb)$, that is $\mathrm{tp}_\exists(a'/Cb) \supseteq \mathrm{tp}_\exists(a/Cb)$. On the other hand, Remark II.2.5 says that $\mathrm{tp}_\exists(a/Cb)$ is a maximal existential type, so $\mathrm{tp}_\exists(a'/Cb) = \mathrm{tp}_\exists(a/Cb)$.

For the "furthermore" part, we can assume that $\phi(x, y, c) \vdash q(x, y, c) = 0 \wedge \exists x' q(x', y, c) \neq 0$, because for $y = b$ we know it is true, so we can take the conjunction of this formula with $\phi(x, y, c)$ without changing $\phi(x, b, c)$. Thus, if $\models \phi(a', b', c)$, then in particular a' is a root of the non-zero polynomial $q(x, b', c)$, so $a' \in \overline{C(b')}$. □

Theorem II.4.8. *Suppose T is a JEP-refinement of $F_{R\text{-module}}$, then $\mathcal{EC}(T)$ has $NSOP_1$.*

Proof. We will use Fact II.2.20, with the weak independence $\mathop{\smile\!\!\!\!\!\!|}^w$.

Let $\mathbb{M}$ be a monster model of T, and let $P = P_\mathbb{M}$. Invariance, symmetry and existence over models are trivial. For monotonicity, suppose $A, B, C, D \subseteq \mathbb{M}$, and $A \mathop{\smile\!\!\!\!\!\!|}^w_C BD$. By monotonicity of independence in ACF, we have $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{ACF}}_C B$. We also get that

$$\begin{aligned} P \cap (\overline{AC} + \overline{BC}) &= P \cap (\overline{AC} + \overline{BDC}) \cap (\overline{AC} + \overline{BC}) \\ &= (P \cap \overline{AC} + P \cap \overline{BDC}) \cap (\overline{AC} + \overline{BC}) \\ &= P \cap \overline{AC} + P \cap \overline{BDC} \cap (\overline{AC} + \overline{BC}) \\ &= P \cap \overline{AC} + P \cap (\overline{BDC} \cap \overline{AC} + \overline{BC}) \end{aligned}$$

where the last two equalities follow from Lemma II.3.1, because $P \cap \overline{AC} \subseteq \overline{AC} + \overline{BC}$ and $\overline{BC} \subseteq \overline{BDC}$. However, $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{ACF}}_C BD$ implies that $\overline{BDC} \cap \overline{AC} = \overline{C}$, so we get $P \cap (\overline{AC} + \overline{BC}) = P \cap \overline{AC} + P \cap \overline{BC}$. Thus, $A \mathop{\smile\!\!\!\!\!\!|}^w_C B$.

Proposition A.2.1 will give us the independence theorem. Note that to use Proposition A.2.1 we need 3-amalgamation of $\mathcal{EC}(T)$, but Lemma II.4.6 gives us 3-amalgamation of $\mathrm{ACF}_{R\text{-module}}$. However, if we start with a weakly independent $\mathcal{P}^-(3)$-system $\{A_s\}_{s\in\mathcal{P}^-(3)}$ of $\mathcal{EC}(T)$, we can turn it into a system of $\mathrm{ACF}_{R\text{-module}}$ by taking the algebraic closure $\overline{A_s}$. The system $\{\overline{A_s}\}_{s\in\mathcal{P}^-(3)}$ is still weakly independent because weak independence is algebraic, i.e. $A \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^w_C B$ implies $\overline{A} \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^w_{\overline{C}} \overline{B}$. The completion of the system, $A_3 \models \mathrm{ACF}_{R\text{-module}}$, can be expanded to a model of T, because A_3 is a model of $F_{R\text{-module}} \cup \mathrm{Th}_\exists(A_\emptyset)$, which is a companion of T (Fact II.2.10).

For strong finite character, suppose $a \not\mathop{\smile\!\!\!\!\!\!\mid}\nolimits^w_M b$, and let $A = \overline{M(a)}$, $B = \overline{M(b)}$. If $a \not\mathop{\smile\!\!\!\!\!\!\mid}\nolimits^{\mathrm{ACF}}_M b$, then the result follows from strong finite character in ACF. Else, there is some $s \in P \cap (A + B) \setminus (P \cap A + P \cap B)$. There are $\alpha \in A$, $\beta \in B$ such that $s = \alpha + \beta$. We claim that $\beta \notin M + P \cap B$. Otherwise, there are some $m \in M$, $p \in P \cap B$ such that $\beta = m + p$, and so $s = \alpha + m + p$. This implies that $s - p = \alpha + m \in P \cap A$, thus $s = \alpha + m + p \in P \cap A + P \cap B$, a contradiction.

There are formulas $\psi_\alpha(y, a, m) \in \mathrm{tp}_\exists(\alpha/Ma)$ and $\psi_\beta(z, b, m) \in \mathrm{tp}_\exists(\beta/Mb)$ isolating their respective types as in Lemma II.4.7. Let $\lambda(x, b, m)$ be the formula

$$\exists y \exists z \psi_\alpha(y, x, m) \wedge \psi_\beta(z, b, m) \wedge y + z \in P,$$

we have $\lambda(x, b, m) \in \mathrm{tp}_\exists(a/Mb)$.

Suppose that $a' \models \lambda(x, b, m)$, and assume towards contradiction that $a' \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^w_M b$. Let $A' = \overline{M(a')}$, from $a' \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^{\mathrm{ACF}}_M b$ we get $A' \cap B = M$. Let α', β' witness the existence in $\lambda(a', b, m)$, that is $\alpha' \models \psi_\alpha(y, a', m)$, $\beta' \models \psi_\beta(z, b, m)$, and $s' := \alpha' + \beta' \in P$. We have $\beta' \equiv^\exists_{Mb} \beta$, in particular $\beta' \in B$ and $\beta' \notin M + P \cap B$, and by the "furthermore" part of Lemma II.4.7, we can assume that $\alpha' \in A'$. By weak independence, $P \cap (A' + B) = P \cap A' + P \cap B$, so there are $\alpha'' \in P \cap A'$, $\beta'' \in P \cap B$ such that $s' = \alpha'' + \beta''$. It follows that $\alpha'' - \alpha' = \beta' - \beta'' \in A' \cap B = M$, but then $\beta' = \alpha'' - \alpha' + \beta'' \in M + (B \cap P)$, a contradiction. □

Remark II.4.9. *In a recent paper [DK21], Dobrowolski and Kamsma generalized the notion of Kim-independence to positive logic, which is a more general context than the one we deal with. They also prove that the independence relation defined on exponential fields in [HK21b] to prove NSOP*$_1$ *is actually Kim-independence. A similar strategy as the one in [DK21, §10.2] seems to yield that* $\mathop{\smile\!\!\!\!\!\!\mid}\nolimits^w$ *is Kim-independence: extension and transitivity of* $\mathop{\smile\!\!\!\!\!\!\mid}\nolimits^w$ *are similar to [dE21a, Theorem 1.4] and the fact that* $F_{R\text{-module}}$ *is Hausdorff follows from Theorem II.3.9.*

II.5 Higher amalgamation of strong independence

In the previous section, we used the weak independence defined in [dE21b, Definition 3.1]. In the above cited definition, another independence called strong independence was defined. Strong independence is less useful for us in the study of NSOP_1, because the proof of strong finite character does not work for strong independence, yet it still has properties worth studying. In this section we will prove that strong independence has n-amalgamation for every $n \geq 3$.

Note that in [HK21b], Haykazyan and Kirby defined a single independence relation that had both strong finite character and n-amalgamation. This does not seem to be the case in our situation.

Definition II.5.1. For a model $M \models \mathrm{ACF}_{R\text{-module}}$, and subsets $A, B, C \subseteq M$, say that A and B are *strongly independent* over C, and denote $A \mathop{\smile\!\!\!\!\!\!|}^{s}_{C} B$, if $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{ACF}}_{C} B$ and $P_M \cap \overline{ABC} = P_M \cap \overline{AC} + P_M \cap \overline{BC}$.

Lemma II.5.2. *The following are a few model theoretic properties of strong independence that we will use.*

- *(Algebraicity) If $A \mathop{\smile\!\!\!\!\!\!|}_{C} B$, then $\overline{AC} \mathop{\smile\!\!\!\!\!\!|}^{s}_{\overline{C}} \overline{BC}$.*
- *(Monotonicity) If $A \mathop{\smile\!\!\!\!\!\!|}^{s}_{C} BD$, then $A \mathop{\smile\!\!\!\!\!\!|}_{C} B$.*

Proof. Algebraicity is obvious from the definition, and from algebraicity of $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{ACF}}$. For monotonicity, suppose $A \mathop{\smile\!\!\!\!\!\!|}^{s}_{C} BD$, from monotonicity of $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{ACF}}$ we have $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{ACF}}_{C} B$. We also get that

$$\begin{aligned} P \cap (\overline{ABC}) &= P \cap \overline{ABDC} \cap \overline{ABC} \\ &= (P \cap \overline{AC} + P \cap \overline{BDC}) \cap \overline{ABC} \\ &= P \cap \overline{AC} + P \cap \overline{BDC} \cap \overline{ABC} \\ &= P \cap \overline{AC} + P \cap \overline{BC}, \end{aligned}$$

where the second equality is from the definition of strong independence, the third equality is from Lemma II.3.1, and the last equality is from $AB \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{ACF}}_{BC} BDC$, which we get from base monotonicity of $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{ACF}}$. □

Notation II.5.3. *A subset $I \subseteq \mathcal{P}(n)$ is called* downward-closed *if $a \in I$ and $b \subseteq a$ imply $b \in I$.*

For a $\mathcal{P}(n)$ ($\mathcal{P}^-(n)$)-system F of fields with a sub-R-module, and $I \subseteq \mathcal{P}(n)$ ($\mathcal{P}^-(n)$) non-empty downward-closed, let

$$F_I = \overline{\bigcup_{a \in I} F_a}$$
$$P_I = \sum_{a \in I} P_{F_a}$$

Also let $F_{\subsetneq a} = F_{\mathcal{P}^-(a)}$, and the same for P.

Lemma II.5.4. *A $\mathcal{P}(n)$ ($\mathcal{P}^-(n)$)-system M of $\mathrm{ACF}_{R\text{-module}}$ is strongly independent iff it is $\mathop{\smile\!\!\!\!\!\!|}^{ACF}$-independent and for every $a \in \mathcal{P}(n)$ ($\mathcal{P}^-(n)$) and $I \subseteq \mathcal{P}(a)$ non-empty downward-closed (if $a \in I$ and $b \subseteq a$, then $b \in I$), we have $P_a \cap M_I = P_I$.*

Proof. For the left to right direction, suppose M is strongly independent. The proof is by induction on $|I|$. If $|I| = 1$, then we must have $I = \{\emptyset\}$, so this case is trivial as $P_a \cap M_\emptyset = P_\emptyset$. If $|I| > 1$, then take a maximal $b \in I$ and let $I' = I \setminus \{b\}$, which is also non-empty downward-closed. By strong independence, $M_b \mathop{\smile\!\!\!\!\!\!|}^{s}_{M_{\subsetneq b}} \bigcup_{b \not\subseteq c \subseteq a} M_c$. For every $c \in I'$ we have $b \not\subseteq c \subseteq a$, so by monotonicity and algebraicity (Lemma II.5.2) $M_b \mathop{\smile\!\!\!\!\!\!|}^{s}_{M_{\subsetneq b}} M_{I'}$. It follows that

$$\begin{aligned} P_a \cap M_I &= P_a \cap \overline{M_b M_{I'}} \\ &= (P_a \cap M_b) + (P_a \cap M_{I'}) \\ &= P_b + P_{I'} = P_I, \end{aligned}$$

where the second equality is from strong independence and the third equality is from the induction assumption.

For the right to left direction, we need to prove $M_b \mathop{\smile\!\!\!\!\!\!\;|}^{s}_{M_{\subsetneq b}} \bigcup_{b\not\subseteq c\subseteq a} M_c$. Consider the downward-closed families $I' = \{c \mid b \not\subseteq c \subseteq a\}$ and $I = I' \cup \{b\}$. With this notation, We need to prove $M_b \mathop{\smile\!\!\!\!\!\!\;|}^{s}_{M_{\subsetneq b}} M_{I'}$. By the assumption,

$$\begin{aligned} P_a \cap \overline{M_b M_{I'}} &= P_a \cap M_I = P_I = P_b + P_{I'} \\ &= (P_a \cap M_b) + (P_a \cap M_{I'}), \end{aligned}$$

and we already know $M_b \mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{ACF}}_{M_{\subsetneq b}} M_{I'}$, so this finishes the proof. □

Theorem II.5.5 (n-amalgamation)**.** *Any strongly independent $\mathcal{P}^-(n)$-system of $\mathrm{ACF}_{R\text{-module}}$ can be completed to a strongly independent $\mathcal{P}(n)$-system.*

Proof. Suppose $M = (M_a)_{a\in\mathcal{P}^-(n)}$ is a strongly independent $\mathcal{P}^-(n)$-system of $\mathrm{ACF}_{R\text{-module}}$. By Proposition A.1.3, there is a field M_n completing M as an independent system of ACF. Define $P_n := P_{\subsetneq n} = \sum_{s\subsetneq n} P_s$, we need to show that (M_n, P_n) completes a strongly independent system. For this we will need the following claim:

Claim. *For every $I, J \subseteq \mathcal{P}(n)$ non-empty downward-closed, $M_I \cap P_J \subseteq P_I$.*

Suppose we proved this claim. For every $a \in \mathcal{P}(n)$, if we take $I = \mathcal{P}(a)$ and $J = \mathcal{P}(n)$, then we will get $M_a \cap P_n \subseteq P_a$, and the other inclusion is obvious, so (M_n, P_n) completes a system of $\mathrm{ACF}_{R\text{-module}}$. Taking $J = \mathcal{P}(a)$ and $I \subseteq \mathcal{P}(a)$, we'll get $M_I \cap P_a \subseteq P_I$, and again the other inclusion is obvious, so by Lemma II.5.4 the system is strongly independent. All that remains is proving the claim.

We will prove the claim by induction on $|IJ|$. The base case is $|IJ| = 1$, which must mean $I = J = \{\emptyset\}$, which is trivial. In the general case, first notice that if $J = \mathcal{P}(n)$, then $P_J = P_n = P_{\mathcal{P}^-(n)}$, so without loss of generality we can assume $J \subseteq \mathcal{P}^-(n)$. If $J \subseteq I$, then it is also trivial, else take some maximal $c \in J$ such that $c \notin I$, and consider $J' = J \setminus \{c\}$, which is also non-empty downward-closed.

We have $P_J = P_c + P_{J'}$, so we need to prove that $M_I \cap (P_c + P_{J'}) \subseteq P_I$. Suppose $p_c + p_{J'} \in M_I \cap (P_c + P_{J'})$ for $p_c \in P_c$, $p_{J'} \in P_{J'}$. In particular, $p_c \in M_{IJ'}$. Remember that $M_{IJ'} = \overline{\bigcup_{a\in IJ'} M_a}$, so there is a tuple $m \in \bigcup_{a\in IJ'} M_a$ such that $q(p_c, m) = 0$ for some non-zero polynomial $q(x, m)$. By Fact II.2.12, there is a tuple $m' \in \bigcup_{a\in IJ'} M_{a\cap c}$ such that $q(p_c, m') = 0$ and $q(x, m')$ is a non-zero polynomial. Let $K = \{a \in IJ' \mid a \subseteq c\} = \{a \cap c \mid a \in IJ'\}$, we get that $p_c \in M_K$. By Lemma II.5.4, $P_c \cap M_K = P_K \subseteq P_{IJ'}$, so $p_c \in P_{IJ'}$. Also, $p_{J'} \in P_{J'} \subseteq P_{IJ'}$, so $p_c + p_{J'} \in P_{IJ'}$. We know that $c \notin IJ'$, so in particular $|IJ'| < |IJ|$, and by the induction hypothesis $M_I \cap P_{IJ'} \subseteq P_I$. Thus, $p_c + p_{J'} \in P_I$, as needed. □

Appendix A

Results on higher amalgamation

Our definition of independent systems, which we borrowed from [HK21b], is not the same the one used by other authors, e.g. [Hru98, dPKM06, GKK13]. It follows that our notion of n-amalgamation is different from the one used in those papers, and adapting results from one definition to another is not trivial. In this appendix we prove well known results about higher amalgamation, using our definition.

A.1 Higher amalgamation of ACF

Under the common definition, ACF has n-amalgamation for every n. More generally, [dPKM06, Proposition 1.6] proves that any stable theory has n-amalgamation over a model for all n. In this section we prove that ACF has n-amalgamation per our definition.

First, recall that for fields A, B, C such that $C \subseteq A \cap B$, we say that A *is linearly disjoint from* B *over* C, and denote $A \mathop{\smile\!\!\!\!\!\vert}^{l}_{C} B$, if whenever $a_0, \ldots, a_{n-1} \in A$ are linearly independent over C they are also linearly independent over B. Equivalently, A is linearly disjoint from B over C iff the canonical map $A \otimes B \to A[B]$ is an isomorphism. In particular, if $A \mathop{\smile\!\!\!\!\!\vert}^{l}_{C} B$ and we have maps $f : A \to K$ and $g : B \to K$ (for some field K) such that $f|_C = g|_C$, then we can jointly extend them to a map $A.B \to K$. For more information, see Section I.2 or [Lan72, §III.1.a].

Lemma A.1.1. *Let $F = \{F_a\}_{a \in \mathcal{P}(n)}$ be an independent $\mathcal{P}(n)$-system of ACF, where all embeddings are subset-inclusions. Suppose $a, b_0, \ldots, b_{k-1} \subseteq n$, then*

$$F_a \mathop{\smile\!\!\!\!\!\vert}^{l}_{F_{a\cap b_0} \ldots F_{a \cap b_{k-1}}} F_{b_0} \ldots F_{b_{k-1}}.$$

Proof. Suppose $\sum_i \alpha_i \beta_i = 0$ for $\alpha_i \in F_a$ and $\beta_i \in F_{b_0} \ldots F_{b_{k-1}}$. We can write $\beta_i = q_i(\beta_{i,0}, \ldots, \beta_{i,k-1})$ for $\beta_{i,j} \in F_{b_j}$ and q_i a rational function. By Fact II.2.12, there exist $\gamma_{i,j} \in F_{a \cap b_j}$ such that $\sum_i \alpha_i q_i(\gamma_{i,0}, \ldots, \gamma_{i,k-1}) = 0$. Denote

$$\gamma_i = q_i(\gamma_{i,0}, \ldots, \gamma_{i,k-1}) \in F_{a\cap b_0} \ldots F_{a \cap b_{k-1}},$$

we have $\sum_i \alpha_i \gamma_i = 0$ as needed. $\square$

Lemma A.1.2. *Let $F = \{F_a\}_{a \in \mathcal{P}(n)}$ ($n > 0$) be an independent $\mathcal{P}(n)$-system of ACF, where all embeddings are subset-inclusions. Suppose K is another field, and for every $a \subsetneq n$ there is an embedding $\tau_a : F_a \to K$, such that $\tau_a|_{F_b} = \tau_b$ for $b \subseteq a \subsetneq n$. Furthermore, suppose that T is a transcendence basis of F_n over $\bigcup_{a \subsetneq n} F_a$ and that $S \subseteq K$ is algebraically independent over $\bigcup_{a \subsetneq n} \tau_a(F_a)$ with $|S| = |T|$. Then there exists an embedding $\tau_n : F_n \to K$ such that $\tau_n|_{F_a} = \tau_a$ for $a \subseteq n$ and $\tau_n(T) = S$.*

Proof. For $i, j < n$, denote $\widehat{i} = n \setminus \{i\}$ and $\widehat{i,j} = n \setminus \{i,j\}$. We will build by induction maps $\sigma_m : F_{\widehat{0}} \dots F_{\widehat{m-1}} \to K$ such that $\sigma_m|_{F_{\widehat{i}}} = \tau_{\widehat{i}}$ for $i < m \leq n$. For $m = 1$, set σ_1 to be $\tau_{\widehat{0}}$. Suppose we defined σ_m, by Lemma A.1.1

$$F_{\widehat{m}} \underset{F_{\widehat{0,m}} \dots F_{\widehat{m-1,m}}}{\overset{l}{\downarrow}} F_{\widehat{0}} \dots F_{\widehat{m-1}}.$$

Furthermore, for every $i < m$

$$\tau_{\widehat{m}}|_{F_{\widehat{i,m}}} = \tau_{\widehat{i,m}} = \tau_{\widehat{i}}|_{F_{\widehat{i,m}}} = \sigma_m|_{F_{\widehat{i,m}}},$$

so $\tau_{\widehat{m}}$ and σ_m coincide on the base of the independence. Thus, there exists a map $\sigma_{m+1} : F_{\widehat{0}} \dots F_{\widehat{m}} \to K$ such that $\sigma_{m+1}|_{F_{\widehat{0}} \dots F_{\widehat{m-1}}} = \sigma_m$ and $\sigma_{m+1}|_{F_{\widehat{m}}} = \tau_{\widehat{m}}$.

Once we built σ_m for every $1 \leq m \leq n$, extend $\sigma_n : F_{\widehat{0}} \dots F_{\widehat{n-1}} \to K$ to an embedding $\tau_n : F_n \to K$ by mapping T to S and extending to the algebraic closure. $\square$

Proposition A.1.3. *ACF has n-amalgamation for every n, with respect to non-forking independence.*

Proof. Let $F = \{F_a\}_{a \in \mathcal{P}^-(n)}$ be an independent $\mathcal{P}^-(n)$-system of ACF with embeddings $\tau_{b,a} : F_b \to F_a$ for $b \subseteq a$. For every $\emptyset \subsetneq a \subsetneq n$, let T_a be a transcendence basis of F_a over $\bigcup_{b \subsetneq a} \tau_{b,a}(F_b)$. By induction on $|a|$, it follows that

$$F_a = \overline{\tau_{\emptyset,a}(F_\emptyset)(\bigcup_{\emptyset \subsetneq b \subseteq a} \tau_{b,a}(T_b))}.$$

Let F_n be some algebraically closed field extension of $F_\emptyset$, with a large enough transcendence degree over $F_\emptyset$. Let $\{S_a\}_{\emptyset \subsetneq a \subsetneq n}$ be some disjoint family of subsets of F_n such that $|S_a| = |T_a|$ and $\bigcup_{\emptyset \subsetneq a \subsetneq n} S_a$ is algebraically independent over $F_\emptyset$. We will extend F to a $\mathcal{P}(n)$-system by defining embeddings $\tau_{a,n} : F_a \to F_n$ for all $a \subsetneq n$. The embeddings $\tau_{a,n}$ will be built by induction on $|a|$.

For $a = \emptyset$, define $\tau_{\emptyset,n} : F_\emptyset \to F_n$ to be the inclusion map. For $a \neq \emptyset$, suppose we built $\tau_{b,n}$ for every $b \subsetneq a$. Consider $\{\tau_{b,a}(F_b)\}_{b \subseteq a}$ as an independent $\mathcal{P}(a)$-system (where the embeddings are subset-inclusions). By Lemma A.1.2, there exist an embedding $\tau_{a,n} : F_a \to F_n$ such that $\tau_{a,n} \circ \tau_{b,a} = \tau_{b,n}$ for $b \subsetneq a$ and $\tau_{a,n}(T_a) = S_a$.

This completes F to a $\mathcal{P}(n)$-system, it remains to prove independence. Consider all $\{F_a\}_{a \subsetneq n}$ as subsets of F_n by taking their image under $\tau_{a,n}$. Notice that by the way we defined $\tau_{a,n}$ (specifically, because $\tau_{\emptyset,n}(F_\emptyset) = F_\emptyset$ and $\tau_{a,n}(T_a) = S_a$), we have that after taking the image under $\tau_{a,n}$

$$F_a = \overline{F_\emptyset(\bigcup_{\emptyset \subsetneq b \subseteq a} S_b)}.$$

We need to prove that for every $a \subseteq n$

$$F_a \underset{\bigcup_{b \subsetneq a} F_b}{\overset{\mathrm{ACF}}{\mathop{\smile\!\!\!\!\!\!\mid}}} \bigcup_{a \not\subseteq c \subseteq n} F_c,$$

which is the same, up to taking algebraic closures, as

$$F_\emptyset\Big(\bigcup_{\emptyset \subsetneq d \subseteq a} S_d\Big) \underset{F_\emptyset(\bigcup_{\emptyset \subsetneq b \subsetneq a} S_b)}{\overset{\mathrm{ACF}}{\mathop{\smile\!\!\!\!\!\!\mid}}} F_\emptyset\Big(\bigcup_{a \not\subseteq c \subseteq n} S_c\Big).$$

This follows from the fact that S_a is algebraically independent over $F_\emptyset(\bigcup_{a \not\subseteq c \subseteq n} S_c)$. □

A.2 The independence theorem

It is a well known fact in the folklore that the independence theorem is equivalent to 3-amalgamation. In our case there are two differences, the definition of 3-amalgamation is different and we work in the category of existentially closed models. We reprove this equivalence in our setting.

Proposition A.2.1. *Let $\mathbb{M}$ be a monster model of an inductive theory T with JEP. Suppose that there is an ternary relation $\mathop{\smile\!\!\!\!\!\!\mid}$ on subsets of $\mathbb{M}$ satisfying invariance, existence, monotonicity, symmetry, and extension. For $M \in \mathcal{EC}(T)$, the following are equivalent:*

1. *(3-amalgamation) every independent $\mathcal{P}^-(3)$-system of $\mathcal{EC}(T)$ over M can be completed to an independent $\mathcal{P}(3)$-system of $\mathcal{EC}(T)$ (a system F is over M if $F_\emptyset = M$).*
2. *(strengthened independence theorem) for tuples c_1, c_2, b_1, b_2 such that $c_1 \mathop{\smile\!\!\!\!\!\!\mid}_M c_2$, $b_1 \mathop{\smile\!\!\!\!\!\!\mid}_M c_1$, $b_2 \mathop{\smile\!\!\!\!\!\!\mid}_M c_2$ and $b_1 \equiv^\exists_M b_2$, there exists b such that $b \equiv^\exists_{Mc_1} b_1$, $b \equiv^\exists_{Mc_2} b_2$, and $b \mathop{\smile\!\!\!\!\!\!\mid}_M c_1c_2$, $bc_1 \mathop{\smile\!\!\!\!\!\!\mid}_M c_2$, $bc_2 \mathop{\smile\!\!\!\!\!\!\mid}_M c_1$.*

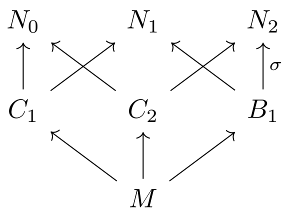
Proof. (1) $\implies$ (2): We can find existentially closed models $C_1, C_2 \in \mathcal{EC}(T)$ such that $Mc_i \subseteq C_i$ $(i = 1, 2)$ and $C_1 \mathop{\smile\!\!\!\!\!\!\mid}_M C_2$ — start with some $Mc_1 \subseteq C_1 \in \mathcal{EC}(T)$, and using extension and invariance move it by an automorphism fixing Mc_1 so that $C_1 \mathop{\smile\!\!\!\!\!\!\mid}_M c_2$, then do the same with some $Mc_2 \subseteq C_2 \in \mathcal{EC}(T)$. By extension, we can find $b'_i \equiv^\exists_{Mc_i} b_i$ $(i = 1, 2)$ such that $b'_i \mathop{\smile\!\!\!\!\!\!\mid}_M C_i$. Note that

$$b'_1 \equiv^\exists_M b_1 \equiv^\exists_M b_2 \equiv^\exists_M b'_2.$$

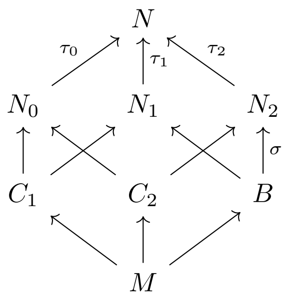
We can find existentially closed models $Mb'_i \subseteq B_i \in \mathcal{EC}(T)$ such that $B_1 \equiv^\exists_M B_2$ and $B_i \mathop{\smile\!\!\!\!\!\!\mid}_M C_i$ $(i = 1, 2)$ — start with some $Mb'_1 \subseteq B_1 \in \mathcal{EC}(T)$, as before use extension and invariance to assume $B_1 \mathop{\smile\!\!\!\!\!\!\mid}_M C_1$, then let B_2 be the image of B_1 under an automorphism given by $b'_1 \equiv^\exists_M b'_2$, by extension and invariance we can move B_2 by an automorphism fixing Mb'_2 such that $B_2 \mathop{\smile\!\!\!\!\!\!\mid}_M C_2$.

Let $N_0, N_1, N_2 \subseteq \mathbb{M}$ be some existentially closed models such that $C_1, C_2 \subseteq$

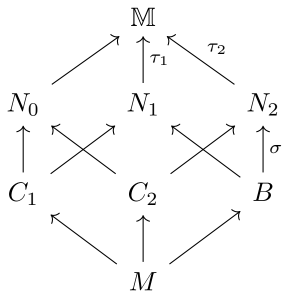
N_0, $C_1, B_1 \subseteq N_1$ and $C_2, B_2 \subseteq N_2$, and consider the $\mathcal{P}^-(3)$-system



where all the arrows are inclusions, except for σ which maps B_1 to B_2, fixing M. The above system is independent, so it can be completed to an independent $\mathcal{P}(3)$-system



We can expand N to the monster $\mathbb{M}$, and by Remark II.2.16 we can expand τ_0, τ_1, τ_2 to automorphisms of $\mathbb{M}$. By applying τ_0^{-1} to $\mathbb{M}$, we can assume that τ_0 is the identity



Let $b = \tau_1(b_1') = \tau_2(b_2')$. By following the diagram, we see that τ_1 fixes Mc_1 and τ_2 fixes Mc_2, so

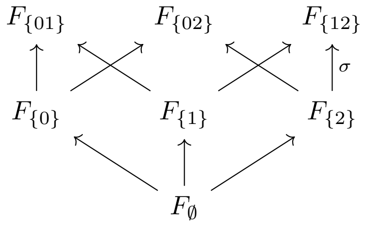
$$b \equiv^{\exists}_{Mc_1} b_1' \equiv^{\exists}_{Mc_1} b_1,$$

$$b \equiv^{\exists}_{Mc_2} b_2' \equiv^{\exists}_{Mc_2} b_2.$$

The independences we need to show follow from the fact that the system is independent (using monotonicity).

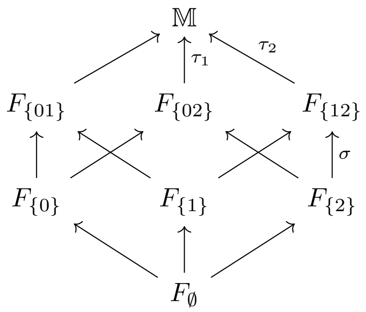
(2) $\implies$ (1): For the other direction, let F be an independent $\mathcal{P}^-(3)$-system over M. We will show that all but one of the embeddings can be assumed to be

inclusions:



Start by replacing $F_\emptyset, F_{\{0\}}, F_{\{1\}}$ with their images in $F_{\{01\}}$. Now move $F_{\{02\}}$ so that the embedding $F_{\{0\}} \to F_{\{02\}}$ would be an inclusion (the system stays independent by invariance), and replace $F_{\{2\}}$ with its image in $F_{\{02\}}$. Finally, move $F_{\{12\}}$ so that the embedding $F_{\{1\}} \to F_{\{12\}}$ would be an inclusion. We are left only with $F_{\{2\}} \xrightarrow{\sigma} F_{\{12\}}$, which we can't assume to be an inclusion.

Recall that $M = F_\emptyset$, and consider $c_1 = F_{\{0\}}$, $c_2 = F_{\{1\}}$, $b_1 = F_{\{2\}}$ and $b_2 = \sigma(b_1)$ as tuples. The conditions for the independence theorem hold from the independent system, so there is some b satisfying $b \equiv_{Mc_1} b_1$, $b \equiv_{Mc_2} b_2$, $b \mathop{\smile\hspace{-0.9em}\mid\hspace{0.9em}}_M c_1c_2$, $bc_1 \mathop{\smile\hspace{-0.9em}\mid\hspace{0.9em}}_M c_2$ and $bc_2 \mathop{\smile\hspace{-0.9em}\mid\hspace{0.9em}}_M c_1$. There are automorphisms τ_1, τ_2 such that $\tau_1 : b_1c_1 \mapsto bc_1$, $\tau_2 : b_2c_2 \mapsto bc_2$, so the following diagram commutes:



We have $b \mathop{\smile\hspace{-0.9em}\mid\hspace{0.9em}}_M c_1c_2$, so by extension, by possibly changing b and thus also changing τ_1, τ_2 while fixing Mc_1c_2, we have $b \mathop{\smile\hspace{-0.9em}\mid\hspace{0.9em}}_M F_{\{01\}}$, which is $F_{\{2\}} \mathop{\smile\hspace{-0.9em}\mid\hspace{0.9em}}_{F_\emptyset} F_{\{01\}}$. We also know $bc_1 \mathop{\smile\hspace{-0.9em}\mid\hspace{0.9em}}_M c_2$, so by extension, possibly changing τ_1, we get $F_{\{02\}} \mathop{\smile\hspace{-0.9em}\mid\hspace{0.9em}}_{F_\emptyset} F_{\{1\}}$. Similarly, we get $F_{\{12\}} \mathop{\smile\hspace{-0.9em}\mid\hspace{0.9em}}_{F_\emptyset} F_{\{0\}}$. Next, by existence, $F_{\{0\}}F_{\{1\}} \mathop{\smile\hspace{-0.9em}\mid\hspace{0.9em}}_{F_{\{0\}}F_{\{1\}}} F_{\{02\}}F_{\{12\}}$, so by extension and changing $F_{\{01\}}$ (really, its embedding into $\mathbb{M}$) we get that $F_{\{01\}} \mathop{\smile\hspace{-0.9em}\mid\hspace{0.9em}}_{F_{\{0\}}F_{\{1\}}} F_{\{02\}}F_{\{12\}}$. The same can be done with $F_{\{02\}} \mathop{\smile\hspace{-0.9em}\mid\hspace{0.9em}}_{F_{\{0\}}F_{\{2\}}} F_{\{01\}}F_{\{12\}}$ and $F_{\{12\}} \mathop{\smile\hspace{-0.9em}\mid\hspace{0.9em}}_{F_{\{1\}}F_{\{2\}}} F_{\{01\}}F_{\{02\}}$. Notice that the automorphisms we take preserve $F_{\{0\}}F_{\{1\}}F_{\{2\}}$, so they preserve the independences already established. This gives us an independent $\mathcal{P}(3)$-system that completes the given independent $\mathcal{P}^-(3)$-system. □

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