Nullstelenzats seminar - proof of the chromatic nullstelenzats

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1 Reminder

This week, we will finish the proof of the chromatic nullstelenzats theorem. Let us first recall the definition and the statement of the theorem

Definition 1.1. A presentable ∞-category $\mathscr{C}$ is called *nullstelenzatsian* if every compact non-terminal $Y \in \mathscr{C}^{\omega}$ admits a map to the initial object. A non-terminal object $X \in \mathscr{C}$ is called *nullstelenzatsian* if $\mathscr{C}_{X/}$ is nullstelenzatsian

Hilbert's nullstelenzats states that the nullstelenzatsian commutative rings are the algebraically closed fields. The chromatic nullstelenzats provides an analog of algebraically closed fields in the chromatic world, in the form of Lubin-Tate rings.

Theorem 1.2. An $\mathbb{E}_\infty$-algebra $R \in \mathrm{CAlg}(\mathrm{Sp}_{T(n)})$ is nullstelenzatsian if and only if $R \simeq E(L)$ for some algebraically closed field L.

Our efforts up to now were in proving the "if" direction - that $E(L)$ is nullstelenzatsian. We will recall the different ingredients, and show how they add up to a proof.

2 $E(L)$ is nullstelenzatsian

Denote $k = \mathbb{F}_p$, $E = E(k)$ and $\mathrm{CAlg}_E = \mathrm{CAlg}_E(\mathrm{Sp}_{T(n)})$.

We defined three nilpotence detecting map $f, g, h \in \mathrm{CAlg}_E$, whose definition we will recall when they come into play, and using a small objects argument argued that every $S \in \mathrm{CAlg}_E$ has some nilpotence detecting map $S \to R$ such that $R \perp f, g, h$.

Recall the adjunction

$$E(-) : \mathrm{Perf}_k \leftrightarrows \mathrm{CAlg}_E : (-)^\flat = \left({}^{\pi_0(-)}/_{m}\right)^\flat$$

We want to prove that the constructed R is in the image of $E(-)$. Due to the fact that $E(-)$ is a fully faithful left adjoint, it is equivalent to the counit $E(R^\flat) \to R$ being an isomorphism.

$\pi_*(E(R^\flat))$ and $\pi_*(R)$ are 2-periodic, so to prove that $\pi_*(E(R^\flat)) \to \pi_*(R)$ is an isomorphism it is enough to check for $* = 0, 1$. However, $\pi_1(E(R^\flat)) = 0$, so we reduce to the following goals

(1) $\pi_1(R) = 0$,

(2) $\pi_0(E(R^\flat)) \to \pi_0(R)$ is injective, and

(3) $\pi_0(E(R^\flat)) \to \pi_0(R)$ is surjective.

We will replace condition (2) by a stronger statement, that $R^\flat$ is of Krull dimension 0.

Lemma 2.1. *Suppose $R^\flat$ is of Krull dimension 0, then $\pi_0(E(R^\flat)) \to \pi_0(R)$ is injective.*

Proof. We will first prove that the reduction modulu m

$$R^\flat \to \pi_0(R)/m$$

is injective. Denote $B = \pi_0(R)/m$, by definition $R^\flat = B^\flat$, so we need to show that $B^\flat \to B$ is injective. Let $x \in \ker(B^\flat \to B)$, because B is of Krull dimension 0, (x) is generated by an idempotent $e \in B^\flat$. e is a system of p^n-th roots of 0 which are also idempotents. The only nilpotent idempotent is 0, so $e = 0$ so $x = 0$

Now consider the following commutative diagram

$$\begin{tikzcd}
\pi_0 E(R^\flat) \arrow[r] \arrow[d, "\sim"'] & \pi_0 R \arrow[d] \\
W_{\mathbb{T}}(R^\flat) \arrow[r, hook] & W_{\mathbb{T}}(\pi_0 R/m)
\end{tikzcd}$$

The left map is an isomorphism from cofreeness, and the bottom map is injective because $W_{\mathbb{T}}$ preserves injectivity, so the top map is injective. ☐

Now we can finish the proof.

Proposition 2.2. *The counit $E(R^\flat) \to R$ is an isomorphism, and moreover $R^\flat$ is of Krull dimension 0.*

Proof. Recall our goals,

(1) $\pi_1(R) = 0$,

(2) $R^\flat$ is of Krull dimension 0, and

(3) $\pi_0(E(R^\flat)) \to \pi_0(R)$ is surjective.

Where condition (2) also implies injectivity. Those three conditions follow from orthogonality to h, f and g respectively.

(1): the map h is defined as

$$h : E\{z^1\} \xrightarrow{z^1 \mapsto 0} E$$

where z^1 is in degree 1. The fact that $h \perp R$ implies that every map $E\{z^1\} \to R$ selecting an element of $\pi_1(R)$ can be factored through 0, i.e. $\pi_0(R) = 0$.

(2): The map f is defined as

$$f : E(k[x^{\frac{1}{p^\infty}}]) \xrightarrow{x \mapsto (x,0)} E(k[x^{\pm\frac{1}{p^\infty}}]) \times E$$

And $f \perp R$ implies that $R^\flat$ is of krull dimension 0. The idea is that every $a \in R^\flat$ corresponds to a map $k[x^{\frac{1}{p^\infty}}] \to R^\flat$, which by orthogonality to f and the $E(-) \dashv (-)^\flat$ adjunction can be lifted to a map $k[x^{\pm\frac{1}{p^\infty}}] \times k \to R^\flat$. Let $b \in R^\flat$ be that image of x^{-1} under this map, it follows that $ab^2 = a$, a condition equivalent to Krull dimension 0.

(3): The map

$$g : E\{z^0\} \to E(k\{z^0\}^\sharp)$$

was defined by Lior. A map $E(k\{z^0\}^\sharp) \to R$ corresponds by cofreeness to a map $E\{z_0\} \to E(R^\flat)$, so $g \perp R$ implies the $\pi_0(E(R^\flat)) \to \pi_0(R)$ is surjective. (The target is a free module so it is conservative, thus nilpotence detecting)

□

The map $S \to R \simeq E(R^\flat)$ we constructed is nilpotence detecting. In particular, if $S \neq 0$, then $R^\flat \neq 0$, and there is some ring map $R^\flat \to F$ to an algebraically closed field F.

Theorem 2.3. Suppose L is an algebraically closed field, then $E(L)$ is nullstelenzatsian.

Proof. Let S be a compact non-zero $E(L)$-algebra, we need to construct a map $S \to E(L)$. By the above discussion, there exists some map $S \to E(F)$ where F is an algebraically closed field. The composition $E(L) \to S \to E(F)$, and the fact that $E(-)$ is fully faithful, exhibit F as an L-algebra, and we can assume $L \subseteq F$.

Now consider the poset of finite subsets of F,

$$\mathscr{P} = \{X \subseteq F \mid |X| < \omega\}$$

and the functor $\mathscr{P} \to \mathrm{Perf}_L$ sending $X \mapsto L[X]^\sharp \subseteq F$, where $L[X]$ is the sub-ring of F generated by L and X. We have that $\mathrm{colim}_{X \in \mathscr{P}} L[X]^\sharp = F$, and thus also $\mathrm{colim}_{X \in \mathscr{P}} E(L[X]^\sharp) = E(F)$ by virtue of $E(-)$ being a left adjoint. However, $\mathscr{P}$ is a filtered poset, and S is compact, so the map $B \to E(F)$ factors through some $B \to E(L[X]^\sharp)$.

We wish to construct a map $L[X]^\sharp \to L$, and since L is perfect it is enough to construct a map $L[X] \to L$. This follows from the classical nullstelenzats, as $L[X]$ is a compact L-algebra. □

3 Nullstelenzatsian is $E(L)$

Classically, the defining property of an algebraically closed field L is that any polynomial $P \in L[x]$ that has a root in some extension of L already has a root in L. This can be expressed as an equivalence of extension problems

$$\begin{tikzcd}
L[x] \arrow[r, "P"] \arrow[dr, "0"'] & L[x] \arrow[d, dashed] & & L[x] \arrow[r, "P"] \arrow[dr, "0"'] & L[x] \arrow[d, dashed] \\
& T \arrow[r, phantom, "\Longleftrightarrow" near start, shift left=4ex] & & & L
\end{tikzcd}$$

This generalizes for nullstelenzatsian objects in an arbitrary category, but we will work only with $\mathrm{Sp}_T n$. For the rest of this section, fix $R \in \mathrm{CAlg}(\mathrm{Sp}_{T(n)})$ nullstelenzatsian.

Lemma 3.1. *For any $0 \neq T \in \mathrm{CAlg}_R$, $W_1, W_2 \in \mathrm{Mod}_R^\omega$, and a map $P : R\{W_1\} \to R\{W_2\}$, the following extension problems are equivalent:*

$$\begin{tikzcd}
R\{W_1\} \arrow[r, "P"] \arrow[dr, "0"'] & R\{W_2\} \arrow[d, dashed] & & R\{W_1\} \arrow[r, "P"] \arrow[dr, "0"'] & R\{W_2\} \arrow[d, dashed] \\
& T & \Longleftrightarrow & & R
\end{tikzcd}$$

Proof. It suffices to prove the left to right implication. Consider the free-forgetful adjunction

$$R\{-\} : \mathrm{Mod}_R \leftrightarrows \mathrm{CAlg}_R : U.$$

Since U preserves filtered colimits, the left adjoint $R\{-\}$ preserves compact objects. In particular, $R\{W_1\}$ and $R\{W_2\}$ are compact in CAlg_R. By assumption, we have a commuting square

$$\begin{tikzcd}
R\{W_1\} \arrow[r, "P"] \arrow[d, "0"'] & R\{W_2\} \arrow[d, dashed] \\
R \arrow[r] & T
\end{tikzcd}$$

and thus a map from the pushout to T

$$Q := R \otimes_{R\{W_1\}} R\{W_2\} \to T$$

where Q is compact in CAlg_R as a pushout of compacts. Moreover, since $T \neq 0$, the existence of $Q \to T$ implies $Q \neq 0$. Thus, since R is nullstelenzatsian, there exists a map $Q \to R$ which gives the desired extension. ☐

Corollary 3.2. *For any $0 \neq T \in \mathrm{CAlg}_R$ and $W \in \mathrm{Mod}_R^\omega$, the induced map on mapping spectra*

$$\mathrm{Map}_{\mathrm{Sp}_{T(n)}}(W, R) \to \mathrm{Map}_{\mathrm{Sp}_{T(n)}}(W, T)$$

is injective on π_*.

Proof. Suppose

$$x \in \ker(\pi_k \mathrm{Map}_{\mathrm{Sp}_{T(n)}}(W, R) \to \pi_k \mathrm{Map}_{\mathrm{Sp}_{T(n)}}(W, T)).$$

As an element of $\pi_k \mathrm{Map}_{\mathrm{Sp}_{T(n)}}(W, R)$, x induces a map of R-algebras $x : R\{\Sigma^k W\} \to R$, and the composition $R\{\Sigma^k W\} \to R \to T$ is homotopic to 0

$$\begin{tikzcd}
R\{\Sigma^k W_1\} \arrow[r, "x"] \arrow[dr, "0"'] & R \arrow[d] \\
& T.
\end{tikzcd}$$

By Lemma 3.1 applied to $W_1 = \Sigma^k W$ and $W_2 = 0$, the map $x : R\{\Sigma^k W\} \to R$ is already homotopic to 0, so $x = 0$ in $\pi_k \mathrm{Map}_{\mathrm{Sp}_{T(n)}}(W, R)$. ☐

Now let $R \in \mathrm{CAlg}(\mathrm{Sp}_{T(n)})$ be nullstelenzatsian. As we saw earlier, there is some algebra map $R \to E(F)$ where F is an algebraically closed field. Let V_n be a type n Smith-Toda complex, since $L_{T(n)}V_n$ is compact in $\mathrm{Sp}_{T(n)}$, the above corollary implies that

$$V_n \otimes R \to V_n \otimes E(F)$$

is injective on π_*. Since $E(F)$ is even, it follows that $V_n \otimes R$ is even. This implies that R is even, and in particular complex orientable.

Since R is even and complex orientable, we have elements $p, v_0, \dots, v_{n-1} \in \pi_* V_n$, and the R-module

$$K := R/(p, v_0, \dots, v_{n-1}) = R/m$$

acquires the structure of an $\mathbb{E}_1$-R-algebra (Hahn-Wilson). Our goal is to show that $R \simeq E(\pi_0 K)$. This will follow from the following proposition:

Proposition 3.3. *$\pi_* K$ is even periodic and $\pi_0 K$ is an algebraically closed field.*

Proof. As an R-module, K is compact as a finite colimit of R, and in particular dualizable. Using its dual R-module $K^\vee$, which is also compact, and the map $R \to E(F)$ in Corollary 3.2, we get that

$$\mathrm{Map}_{\mathrm{Sp}_{T(n)}}(K^\vee, R) \to \mathrm{Map}_{\mathrm{Sp}_{T(n)}}(K^\vee, E(F))$$

is injective on π_*, or equivalently

$$K \to K \otimes_R E(F) = E(F)/m$$

is injective on π_*. Since $\pi_*(E(F)/M) = F[u^{\pm 1}]$ is even and commutative, it follows that $\pi_*(K)$ is even and commutative.

To continue, we will need the following claim: For every $f \in \pi_* K[t_1, \dots, t_l]$ non-constant homogeneous where $|t_i| = 2d_i$, there exist $x_1, \dots, x_l \in \pi_* K$ of degree $|x_i| = 2d_i$ such that $f(x_1, \dots, x_l) = 0$ ($\pi_* K$ is nullstelenzatsian as a graded commutative ring).

First, we will show that $\pi_*(E(F)/m) = F[u^{\pm 1}]$ satisfies this claim. Indeed, given $f \in F[u^{\pm 1}][t_1, \dots, t_l]$ as above, we can reduce to $f \in F[t_1, \dots, t_l]$ and $|t_i| = 0$ by multiplying with u, which follows from the classical nullstelenzats.

Now let $f \in \pi_* K[t_1, \dots, t_l]$ as above, and let $\tilde{f} \in \pi_* K\langle t_1, \dots, t_l \rangle$ be a lift to non-commutative polynomials. Such a polynomial defines a map $\oplus_{i=1}^{l} \Sigma^{-2d_i} K \to \Sigma^{-2d} K$, which has a transpose $P_f : \Sigma^{2d} K^\vee \to \oplus_{i=1}^{l} \Sigma^{2d_i} K^\vee$. The fact that f has a solution in $\pi_*(E(F)/m) = \pi_*(E(F) \otimes_R K)$ corresponds to an extension

$$\begin{tikzcd}
R\{\Sigma^{2d} K^\vee\} \arrow[r, "P_f"] \arrow[dr, "0"'] & R\{\oplus_{i=1}^{l} \Sigma^{2d_i} K^\vee\} \arrow[d, dashed] \\
& E(F).
\end{tikzcd}$$

so by Lemma 3.1 there also exist an extension to R, which corresponds to a solution in $\pi_* K$.

Finally, we will use the above claim with the following polynomials:

(1) $f(t_1, t_2) = t_1 t_2 - 1$ where $|t_1| = 2$, $|t_2| = -2$ implies the existence of an invertible element of degree 2.

(2) $f(t) = at - 2$ where $|t| = 0$, $0 \neq a \in \pi_0 K$ implies that $\pi_0 K$ is a field

(3) Any non-constant $f(t) \in \pi_0 K[t]$ where $|t| = 0$ implies that $\pi_0 K$ is algebraically closed.

□

Corollary 3.4. *The above proposition implies that $R \simeq E(\pi_0 K)$ (Lurie).*

4 The constructible spectrum

Given a commutative ring A, a geometric point of $\mathrm{Spec}(A)$ is a map $\mathrm{Spec}(L) \to \mathrm{Spec}(A)$ for L algebraically closed, where two point are identified if they factor through a common field extension. Moreover, The set of geometric points can be given a topology, called the constructible topology, where the closed sets are images of $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$.

Now that we have a notion of algebraically closed in the chromatic world, we can define a similar spectrum.

Definition 4.1. For $R \in \mathrm{CAlg}(\mathrm{Sp}_T n)$, define $\mathrm{Spec}^{cons}_{T(n)}(R)$ to be the set of maps $q : R \to E(L)$ for L algebraically closed modulo the relation identifying $q_1 : R \to E(L_1)$ and $q_2 : R \to E(L_2)$ if there exists some algebraically closed field L_3 and a square

$$\begin{tikzcd} R \arrow[r, "q_1"] \arrow[d, "q_2"'] & E(L_1) \arrow[d] \\ E(L_2) \arrow[r] & E(L_3), \end{tikzcd}$$

with the topology where the closed sets are the sets of point that factor through some $R \to S$.

Some if the properties we discussed have a geometric interpretation in the constructible spectrum. for example, the fact that every $R \neq 0$ has some map $R \to E(L)$ implies that $R = 0 \iff \mathrm{Spec}^{cons}_{T(n)}(R) \neq$. For another example, we have:

Proposition 4.2. *A map $R \to S$ detects nilpotence if and only if $\mathrm{Spec}^{cons}_{T(n)}(S) \to \mathrm{Spec}^{cons}_{T(n)}(R)$ is surjective.*

Proof. Suppose that $R \to S$ is nilpotence detecting, in particular it is nil-conservative, meaning that for every $0 \neq A \in \mathrm{CAlg}_R$ $A \otimes_R S \neq 0$. For every geometric point $R \to E(L)$, the fiber of this point in $\mathrm{Spec}^{cons}_{T(n)}(S)$ is given by the geometric points of $E(L) \otimes_R S$. Since $E(L) \otimes_R S \neq 0$, the fiber is non-empty.

For the other direction, first note that for any $R \in \mathrm{CAlg}(\mathrm{Sp}_T n)$ there exists a nilpotence detecting map $R \to E(A)$ where A is a product of algebraically closed fields. Indeed, we have $R \to E(A')$ where A' is perfect of Krull dimension 0, and we let A be the product of the algebraic closure of the residue fields of A'.

Now suppose that $\mathrm{Spec}^{cons}_{T(n)}(S) \to \mathrm{Spec}^{cons}_{T(n)}(R)$ is surjective, and choose nilpotence detecting maps $R \to E(A)$ and $E(A) \otimes_R S \to E(B)$ such that A and B are products of algebraically closed fields. Consider the diagram

$$\begin{tikzcd}
R \arrow[r] \arrow[d] & S \arrow[d] & \\
E(A) \arrow[r] & E(A) \otimes_R S \arrow[r] & E(B)t
\end{tikzcd}$$

Since $\mathrm{Spec}^{cons}_{T(n)}(S) \to \mathrm{Spec}^{cons}_{T(n)}(R)$ is surjective it follows that $\mathrm{Spec}^{cons}_{T(n)}(E(A) \otimes_R S) \to \mathrm{Spec}^{cons}_{T(n)}(E(A))$ is surjective, indeed, the fiber of $E(A) \to E(L)$ is given by $E(L) \otimes_R S$

$$\begin{tikzcd}
R \arrow[r] \arrow[d] & S \arrow[d] & \\
E(A) \arrow[r] \arrow[d] & E(A) \otimes_R S \arrow[r] \arrow[d] & E(B) \\
E(L) \arrow[r] & E(L) \otimes_R S &
\end{tikzcd}$$

which is non-empty. Moreover, the map $E(A) \otimes_R S \to E(B)$ is nilpotence detecting so by the former direction $\mathrm{Spec}^{cons}_{T(n)}(E(B)) \to \mathrm{Spec}^{cons}_{T(n)}(E(A) \otimes_R S)$ is surjective. Altogether, the map $\mathrm{Spec}^{cons}_{T(n)}(E(B)) \to \mathrm{Spec}^{cons}_{T(n)}(E(A))$ is surjective. However, this map is identified to $\mathrm{Spec}^{Zar}(B) \to \mathrm{Spec}^{Zar}(A)$, and it follows that $E(A) \to E(B)$ is nilpotence detecting. In particular $R \to E(B)$ is nilpotence detecting, and together with $S \to E(B)$ being nilpotence detecting we finish. $\square$