# Orientations seminar - Hirzebuch signature theorem

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## 1 Signature

Let $M$ be a closed oriented manifold of dimension $4k$. Consider the bilinear form:

$$I_M : H^{2k}(M) \times H^{2k}(M) \to \mathbb{Z}$$
$$c_1, c_2 \mapsto \langle c_1 \cup c_2, [M] \rangle$$

This bilinear form is called the *intersection form*, because under Poincare duality it corresponds to the bilinear form $H_{2k}(M) \times H_{2k}(M) \to \mathbb{Z}$ which, for classes $[C], [D] \in H_{2k}(M)$ represented by submanifolds of dimension $2k$, acts by wiggling them until they are transverse, and counting their (signed) intersection points. $I_M$ vanishes on the torsion subgroup of $H^{2k}(M)$, so it restricts to a bilinear form on the free quotient

$$\overline{I}_M : \mathrm{Free}H^{2k}(M) \times \mathrm{Free}H^{2k}(M) \to \mathbb{Z}$$

and Poincare duality tells us that $\overline{I}_M$ is non-degenerate. This implies that $I_M$ would be non-degenerate after tensoring with $\mathbb{Q}$, or even $\mathbb{R}$.

**Definition 1.1.** Given a real bilinear form $B : V \times V \to \mathbb{R}$, there exists maximal subspaces $V_+, V_- \subseteq V$ on which $B$ is positive/negative definite. The signature of $B$ is defined as $\mathrm{Sign}(M) = \dim(V_+) - \dim(V_-) \in \mathbb{Z}$. Alternatively, given a diagonal representation of $V$, the signature is the number of positive entries minus the number of negative entries.

**Definition 1.2.** For $M$ closed oriented, the signature of $M$ is $\mathrm{Sign}(M) = \mathrm{Sign}(I_M \otimes R) \in \mathbb{Z}$.

**Example 1.3.** Consider $S^2 \times S^2$, $H^2(S^2 \times S^2; \mathbb{R}) \simeq \mathbb{R}^2$. In the standard basis, the intersection form is represented by a matrix

$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

where $H$ stands for hyperbolic. By diagonalizing $H$, we see that the signature is 0.

**Example 1.4.** For $\mathbb{CP}^{2k}$, the middle homology is $H_{2k}(\mathbb{CP}^{2k}) \simeq \mathbb{Z}$ with generator $[\mathbb{CP}^k]$. The submanifold $\mathbb{CP}^k \subseteq \mathbb{CP}^{2k}$ is a locus of a degree 1 homogeneous polynomial so by Bezout's theorem the intersection number is $I_{\mathbb{CP}^{2k}}(\mathbb{CP}^k, \mathbb{CP}^k) = 1$. Thus, $\mathrm{Sign}(\mathbb{CP}^{2k}) = 1$.

Our first goal is the following theorem:

**Theorem 1.5.** The signature is a genus, i.e. a ring homomorphism $\mathrm{Sign}\colon \Omega^{SO} \to \mathbb{Z}$.

We need to show that Sign is bordism invariant and a ring homomorphism. We will start with the latter.

**Lemma 1.6.** *Let $M_1^{4k_1}, M_2^{4k_2}$ be oriented manifolds, then:*

(1) $\mathrm{Sign}(M_1 \sqcup M_2) = \mathrm{Sign}(M_1) + \mathrm{Sign}(M_2)$ *if* $k_1 = k_2 = k$.

(2) $\mathrm{Sign}(M_1 \times M_2) = \mathrm{Sign}(M_1)\mathrm{Sign}(M_2)$

*Proof.* For additivity, we have $H^{2k}(M_1 \sqcup M_2) = H^{2k}(M_1) \oplus H_{2k}(M_2)$, so the diagonal representation of $I_{M_1 \cup M_2}$ will just be the diagonal representation of $I_{M_1}$ followed by $I_{M_2}$.
For multiplicativity, the Kunneth theorem tells us

$$H^{2k_1+2k_2}(M_1 \times M_2; \mathbb{R}) = \bigoplus_j H^{2k_1+j}(M_1; \mathbb{R}) \otimes H^{2k_2-j}(M_2; \mathbb{R}) = H^{2k_1}(M_1; \mathbb{R}) \otimes H^{2k_2}(M_2; \mathbb{R}) \oplus T$$

We want to prove that the intersection form vanishes on $T$. Note moreover that $[M_1 \times M_2] = [M_1] \otimes [M_2] \in H_{4k_1}(M_1) \otimes H_{4k_2}(M_2)$. ☐

To prove cobordism invariance, we will use the following lemmas:

**Lemma 1.7.** *Let $i\colon M^n \hookrightarrow N^{n+1}$ be the boundary inclusion of oriented manifolds, and let $c \in H^n(N)$. Then $\langle i^*(c), [M] \rangle = 0$*

*Proof.* $\langle i^*(c), [M] \rangle = \langle c, i_*[M] \rangle$ and $i_*[M] = 0$, as it is a boundary in $N$. ☐

**Lemma 1.8.** *Let $B\colon V \times V \to \mathbb{R}$ be a non-degenerate symmetric bilinear form. Suppose $W \subseteq V$ is isotropic, meaning that $B(w_1, w_2) = 0$ for all $w_1, w_2 \in W$, and $\dim(V) = 2\dim(W)$. The* $\mathrm{Sign}(B) = 0$.

*Proof.* Let $0 \neq w_1 \in W_1$. Since $B$ is non-degenerate there exists $v_1 \in V$ such that $B(w_1, v_1) = 1$. Adding a multiple of $w_1$ to $v_1$, we can assume $B(v_1, v_1) = 0$ Let $V = \mathrm{Span}(v_1, w_1) \oplus V_1$ be an orthogonal decomposition. $B|_{\mathrm{Span}(v_1,w_1)} = H$ is the hyperbolic form with signature 0, and $V_1$ satisfies the same hypothesis as $V$. Proceed by induction. ☐

*Proof of theorem.* We want to show that if $M^{4k} \hookrightarrow N^{4k+1}$ is a boundary inclusion, then $\mathrm{Sign}(M) = 0$. Consider the map between long exact sequences induces for Poincare duality:

$$\begin{array}{ccccc} H^{2k}(N;\mathbb{R}) & \xrightarrow{i^*} & H^{2k}(M;\mathbb{R}) & \longrightarrow & H^{2k+1}(N,M;\mathbb{R}) \\ \downarrow\sim & & \downarrow\sim & & \downarrow\sim \\ H_{2k+1}(N,M;\mathbb{R}) & \longrightarrow & H_{2k}(M;\mathbb{R}) & \xrightarrow{i_*} & H_{2k}(N;\mathbb{R}) \end{array}$$

We claim the $\mathrm{Im}(i^*)$ is isotropic for $I_M \otimes \mathbb{R}$ and has dimension $\frac{1}{2}\dim H^{2k}(M;\mathbb{R})$. Isotropy follows from Lemma 1.7. $\mathrm{Im}(i^*)$ maps isomorphically to $\ker(i_*)$, but $\ker(i_*)$ is the orthogonal of $\mathrm{Im}(i^*)$ under the pairing of homology and cohomology, so we deduce $\dim \mathrm{Im}(i^*) = \frac{1}{2}\dim H^{2k}$. ☐

Now that we know that the signature is a genus, we want to be able to it using Pontrjagin numbers. Let's first recall their definition.

## 2 Pontrjagin classes

Consider $c_1(L) = -y \in H^2(\mathbb{CP}^\infty)$ the universal Chern class of the tautological line bundle $L$. Taking complex conjugate is orientation flipping, so $c_1(\overline{L}) = y$.

**Lemma 2.1.** *For every complex vector bundle $E \to M$, $c_i(\overline{E}) = (-1)^i c_i(E)$.*

*Proof.* By the splitting principle, we can think of $c_i(E)$ as the $i$-th elementary symmetric polynomial in $x_1, \ldots, x_k$ where $k$ is the rank of $E$. Each $x_j$ is pulled from $-y \in H^2(\mathbb{CP}^\infty)$, so in $c_i(\overline{E})$ it is replaced with $-x_i$. □

Suppose $V \to M$ is a real vector, then $V \otimes_{\mathbb{R}} \mathbb{C} \to M$ is a complex vector bundle. We have an isomorphism $V \otimes_{\mathbb{R}} \mathbb{C} \simeq \overline{V \otimes_{\mathbb{R}} \mathbb{C}}$, so $c_i(V \otimes_{\mathbb{R}}) = (-1)^i c_i(V \otimes_{\mathbb{R}})$. This implies that the odd Chern classes are 2-torsion.

**Definition 2.2.** The Pontrjagin classes of $V$ are the even Chern classes of $V \otimes_{\mathbb{R}} \mathbb{C}$, up to a sign convention

$$p_i(V) = (-1)^i c_{2i}(V \otimes_{\mathbb{R}} \mathbb{C}) \in H^{4i}(M)$$

The total Pontrjagin class is then $p(V) = 1 + p_1(V) + p_2(V) + \ldots$.

For $E \to M$ a complex vector bundle, we have $E \otimes_{\mathbb{R}} \mathbb{C} \simeq E \oplus \overline{E}$. Writing $c(E) = \prod_{j=1}^k (1 + x_j)$, we have $c(\overline{E}) = \prod_{j=1}^k (1 - x_j)$, and so

$$c(E \oplus \overline{E}) = \prod_{j=1}^k (1 + x_j)(1 - x_j) = \prod_{j=1}^k (1 - x_j^2).$$

It follows that $p_i(E)$ is the $i$-th elementary symmetric polynomial in $x_j^2$. Conversely, every symmetric polynomial in $x_j^2$ formally corresponds to a polynomial in Pontrjagin classes.
Given an oriented $n$-manifold $M$ and a tuple $i_1, \ldots i_r \geq 1$ such that $4(i_1 + \cdots + i_r) = n$, the associated Pontrjagin number is

$$p_{i_1,\ldots,i_r}(M) = \langle p_{i_1}(TM) \cup \cdots \cup p_{i_r}(TM), [M] \rangle$$

**Proposition 2.3.** *The Pontrjagin numbers is oriented-cobordism invariant,*

$$p_{i_1,\ldots,i_r} : \Omega_n^{SO} \to \mathbb{Z}$$

*Proof.* Let $i : M^n \hookrightarrow N^{n+1}$ be a boundary inclusion, we want to show that the Pontrjagin numbers vanish on $M$. Note that $i^*TN = TM \oplus \mathbb{R}$, so from stability $p_i(TM) = i^* p_i(TN)$. The result then follows from the above lemma. □

## 3 The $L$-polynomial

A symmetric polynomial in $x_1^2, \ldots, x_k^2$ corresponds to a unique polynomial in Pontrjagin classes. We can define such polynomials using Taylor series, which will be truncated at the rank. Consider the series

$$\frac{x}{\tanh(x)} = 1 + \frac{x^2}{3} + \ldots$$

and define the *L-polynomial* as $L = \prod_{j=1}^{k} \frac{x_j}{\tanh(x_j)}$. Because $L$ is symmetric, and $\frac{x}{\tanh(x)}$ has only even powers of $x$, we can write $L$ as a polynomial in Pontrjagin classes.

**Definition 3.1.** Given an oriented $n$-manifold $M$, the $L$-genus of $M$ is given by $\langle L(TM), [M]\rangle \in \mathbb{Z}$. Note that when we write this expression, we only multiply the degree $n$ part.

**Proposition 3.2.** *The $L$ genus is indeed a ring homomorphism $\Omega^{SO} \to \mathbb{Z}$.*

*Proof.* Cobordism invariance follows from cobordism invariance of Pontrjagin numbers. Additivity is immediate, as we compute component wise. For multiplicativity, suppose $TM, TM'$ have formal Chern roots $x_1, \dots, x_n$, $x_{n+1}, \dots, x_{n+n'}$ respectively, then

$$L(TM \oplus TM') = \prod_{j=1}^{n+n'} \frac{x_j}{\tanh(x_j)} = L(TM)L(TM').$$

This might formally work only when $M, M'$ have a complex structure and $TM, TM'$ are stably split over $\mathbb{C}$, but in fact this is the only case where we will need it. □

We will calculate the $L$-genus for $\mathbb{CP}^n$. Consider the exact sequence of vector bundles over $\mathbb{CP}^n$:

$$0 \to S \to \underline{\mathbb{C}^{n+1}} \to Q \to 0$$

where $S$ is the tautological line bundle and $Q$ is the quotient. The tangent bundle of $\mathbb{CP}^n$ is given by $T\mathbb{CP}^n = \hom(S, Q) = S^* \otimes Q$.

**Lemma 3.3.** *$T\mathbb{CP}^n$ is stably equivalent to $(S^*)^{n+1}$*

*Proof.* By the exact sequence $\underline{\mathbb{C}^{n+1}} = S \oplus Q$, so

$$(S^*)^n = S^* \otimes \underline{\mathbb{C}^{n+1}} = S^* \otimes S \oplus S^* \oplus Q \simeq \mathbb{C} \oplus T\mathbb{CP}^n$$

□

**Proposition 3.4.** *The $L$-genus of $\mathbb{CP}^n$ is* $\langle L(T\mathbb{CP}^n), [\mathbb{CP}^n]\rangle = \begin{cases} 1 & 2|n \\ 0 & 2 \nmid n \end{cases}$.

*Proof.* By stability of Chern classes, we can replace $T\mathbb{CP}^n$ by $(S^*)^{n+1}$. Note that $(S^*)^{n+1}$ is already a sum of line bundles, so we don't need to split it, and in $(S^*)^{n+1}$ we have $x_j = y \in H^2(\mathbb{CP}^n)$ (it is $y$ and not $-y$ because we are using the dual tautological bundle $S^*$), so

$$L((S^*)^{n+1}) = (\frac{y}{\tanh(y)})^{n+1}$$

We have $\langle y^n, [\mathbb{CP}^n]\rangle = 1$, so the $L$-genus is the coefficient of $y^n$ above, which is equal by the Cauchy integral formula to

$$\frac{1}{2\pi i}\int \frac{dy}{y^{n+1}}(\frac{y}{\tanh(y)})^{n+1} = \frac{1}{2\pi i}\int \frac{dy}{\tanh(y)^{n+1}} =$$
$$\frac{1}{2\pi i}\int \frac{dz}{(1-z^2)z^{n+1}} = \frac{1}{2\pi i}\int \frac{1+z^2+z^4+\dots}{z^{n+1}}dz = \begin{cases} 1 & 2|n \\ 0 & 2 \nmid n \end{cases}$$

□

## 4 Hirzebuch signature theorem

**Theorem 4.1** (Hirzebuch signature theorem)**.** The signature is equal to the $L$-genus

$$\mathrm{Sign}(M) = \langle L(TM), [M] \rangle.$$

Note that both the signature and the $L$-genus are ring homomorphisms $\Omega^{SO} \to \mathbb{Z}$. To check that they are equal it is enough verify after tensoring with $\mathbb{Q}$.

We will use the following result:

**Theorem 4.2.** $\Omega^{SO} \otimes \mathbb{Q} \simeq \mathbb{Q}[y_1, y_2, \dots]$ is a freely generated ring, where the generator $y_i$ of degree $4i$ corresponds to the cobordism class of $\mathbb{CP}^{2i}$.

The Hirzebuch signature theorem then follows, as both the sign and the $L$-genus of $\mathbb{CP}^{2i}$ is 1.

For a 4-manifold, the Hirzebuch signature theorem implies that

$$\mathrm{Sign}(M) = \langle p_1(TM)/3, [M] \rangle$$

In particular, since the left-hand side is an integer, it implies that the first Pontrjagin number of $M$ is divisible by 3. This is not at obvious, as $p_1(TM)/3$ is generally not an integer cohomology class.