The Multiplicative Structure of Higher Bordism Categories

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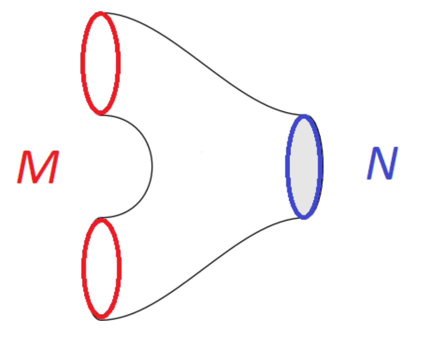
(joint w/ Shai Keidar and Lior Yanovski)

Bordism Rings

Bordism

M, N smooth closed n-manifolds.

Bordism is $n+1$-manifold with boundary $M \sqcup N$



Bordism ring

$$\Omega_n \;=\; {}^{n\text{-manifolds}} \Big/ {}_{\text{bordism}}$$

Ω_* is graded ring:

- $+$: disjoint union
- $\times$: multiplication of manifolds

Bordism ring

Theorem (Thom 1954)

$$\Omega_* \simeq \mathbb{F}_2[x_n \mid n \neq 2^t - 1]$$

Framing

A (stable) framing is a trivialization of the (stable) tangent bundle:

$$\begin{array}{ccc} & & \mathrm{pt} \\ & \nearrow & \downarrow \\ M & \xrightarrow{TM} & BO \end{array}$$

$$\Omega_n^{\mathrm{fr}} = {\text{stably framed } n\text{-manifolds}} \Big/ {\text{stably framed bordism}}$$

Framed bordism ring

Theorem (Pontryagin 1938)

$$\Omega_*^{\mathrm{fr}} \simeq \pi_* \mathbb{S}$$

Higher Bordism Categories

Categorification

Bordism as relation $\rightsquigarrow$ Bordism as structure

category:

- obj: n-manifolds
- mor: bordisms

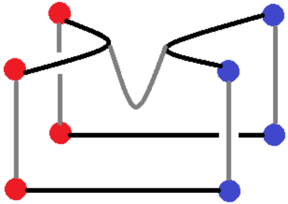
Bordisms between bordisms $\rightsquigarrow$ Higher category

Higher bordism category

Definition (Lurie, Calaque-Scheimbauer)
Bord_n is an (∞, n)-category:

- obj: 0-manifolds
- 1-mor: 1-manifolds with boundary
- 2-mor: 2-manifolds with corners
- $\ldots$ n-mor: n-manifolds with higher corners
- $n+1$-mor: diffeomorphisms
- $\ldots$

Example



Categorifying the ring structure

Disjoint union $\rightsquigarrow$ Symmetric monoidal structure on Bord_n

Multiplication $\rightsquigarrow$?

First attempt

$$\mathrm{Bord}_n \times \mathrm{Bord}_k \stackrel{?}{\to} \mathrm{Bord}_{n+k}$$

Problem!

$$\text{1-mor} \times \text{1-mor} \mapsto \text{1-mor}$$

$$\text{1-manifold} \times \text{1-manifold} \mapsto \textbf{2-manifold}$$

Gray product

Cartesian product:

$$\Delta^1 \times \Delta^1 = \begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ \downarrow & & \downarrow \\ \bullet & \longrightarrow & \bullet \end{array}$$

Gray product:

$$\Delta^1 \overset{\rightarrow}{\times} \Delta^1 = \begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ \downarrow & \swarrow\!\!\!\swarrow & \downarrow \\ \bullet & \longrightarrow & \bullet \end{array}$$

Multiplication of manifolds

Construction
Multiplication of manifolds defines a functor

$$\mathrm{Bord}_n \vec{\times} \mathrm{Bord}_k \to \mathrm{Bord}_{n+k}.$$

induces a symmetric monoidal functor

$$\mathrm{Bord}_n \vec{\otimes} \mathrm{Bord}_k \to \mathrm{Bord}_{n+k}.$$

Highest bordism category

$$\mathrm{Bord}_\infty := \varinjlim \mathrm{Bord}_n$$

Algebra structure:

$$\mathrm{Bord}_\infty \vec{\otimes} \mathrm{Bord}_\infty \to \mathrm{Bord}_\infty$$

Framed bordism category

$\mathrm{Bord}_n^{\mathrm{fr}}$: like Bord_n, everything suitably framed.

$$\mathrm{Bord}_n^{\mathrm{fr}} \vec{\otimes} \mathrm{Bord}_k^{\mathrm{fr}} \to \mathrm{Bord}_{n+k}^{\mathrm{fr}}.$$

Universal property of bordisms

Duality

$\mathcal{C}$ symmetric monoidal $(\infty, 1)$-category. $X \in \mathcal{C}$ is **dualizable** if there exists:

- $X^\vee \in \mathcal{C}$
- $\mathrm{ev}\colon X^\vee \otimes X \to \mathbb{1}$
- $\mathrm{coev}\colon \mathbb{1} \to X \otimes X^\vee$
- zigzag identities

Full duality

$\mathcal{C}$ symmetric monoidal (∞, n)-category. $X \in \mathcal{C}$ is ***n*-fully dualizable** if:

- X is dualizable
- ev and coev have left adjoints
- the units and counits have left adjoints
- ...up to level $n-1$

Bordism Hypothesis

Conjecture (Baez-Dolan, Lurie)
$\mathrm{Bord}_n^{\mathrm{fr}}$ is free on an n-fully dualizable object.

$$F\colon \mathrm{Bord}_n^{\mathrm{fr}} \to \mathcal{C} \quad \leftrightsquigarrow \quad F(\mathrm{pt}) \in \mathcal{C}^{n-\mathrm{fd}}$$

For $n = 1$: Harpaz

Relation to Multiplicative Structure

Theorem (Keidar-Yanovski-N)
The Bordism Hypothesis is equivalent to

$$\mathrm{Bord}_n^{\mathrm{fr}} \vec{\otimes} \mathrm{Bord}_k^{\mathrm{fr}} \xrightarrow{\sim} \mathrm{Bord}_{n+k}^{\mathrm{fr}}.$$

$\implies \mathrm{Bord}_\infty^{\mathrm{fr}}$ is an idempotent algebra.

Proof Idea

Lax Natural Transformations

$\vec{\times}$ has an internal hom $\mathsf{Fun}^{\mathrm{lax}}(\mathcal{C}, \mathcal{D})$:

- functors $F\colon \mathcal{C} \to \mathcal{D}$
- **lax** natural transformations $\alpha\colon F \Rightarrow G$

$$\begin{array}{ccc} FX & \xrightarrow{\alpha_X} & GX \\ {\scriptstyle Ff}\downarrow & \swarrow\!\!\!\!\swarrow & \downarrow{\scriptstyle Gf} \\ FY & \xrightarrow[\alpha_Y]{} & GY \end{array}$$

- …

Lax Natural Transformations

$\vec{\otimes}$ has an internal hom $\mathsf{Fun}^{\mathrm{lax}}_{\otimes}(\mathcal{C}, \mathcal{D})$:

- symmetric monoidal functors
- symmetric monoidal **lax** natural transformations
- …

dbl,R

$\mathcal{C}$ symmetric monoidal (∞, n)-category.

Lemma (Johnson Freyd-Scheimbauer)

$$\mathsf{Fun}^{\mathrm{lax}}_{\otimes}(\mathrm{Bord}^{\mathrm{fr}}_{1}, \mathcal{C}) \simeq \mathcal{C}^{\mathrm{dbl,R}}$$

$$\mathcal{C}^{\mathrm{dbl,R}} \subseteq \mathcal{C}$$

dualizable objects and right adjoint (higher) morphisms.

$\mathcal{C}^{\mathrm{dbl,R}}$ is $(\infty, n-1)$-category.

Iterated dbl,R

$$((\mathcal{C}^{\mathrm{dbl,R}})\cdots)^{\mathrm{dbl,R}} \simeq \mathcal{C}^{n-\mathrm{fd}}$$

$$\Downarrow$$

$$\mathrm{Bord}_1^{\mathrm{fr}} \vec{\otimes} \cdots \vec{\otimes} \mathrm{Bord}_1^{\mathrm{fr}} \simeq \mathrm{Bord}_n^{\mathrm{fr}}$$

Final remarks

- Similar work by Naruki Masuda in categorical spectra.
- Extends to tangle categories.

Thank You!