Rigid Algebras are right adjoint to Cospans

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Motivation - Rigid Categories

Symmetric Monoidal Categories

Symmetric monoidal categories $(\mathcal{C}, \otimes, \mathbb{1})$:

- $(\mathrm{Ab}, \otimes, \mathbb{Z})$
- $(\mathrm{Vect}_k, \otimes_k, k)$

In presentable stable categories:

- $(\mathrm{Sp}, \otimes, \mathbb{S})$
- $(\mathrm{Mod}_R(\mathrm{Sp}), \otimes_R, R)$ for $R \in \mathrm{CAlg}(\mathrm{Sp})$.

"Archetypal" examples, with good properties:

- Dualizable categories
- Dualizable elements = compact elements

Rigid Categories

$(\mathcal{C}, \otimes, \mathbb{1})$ (presentable, stable) is called **rigid** if:

1. $\mathbb{1}$ is compact
2. $\otimes\colon \mathcal{C} \otimes \mathcal{C} \to \mathcal{C}$ is internally left adjoint (in $\mathcal{C}$-bimodules)

For example:

- $\mathrm{Mod}_R(\mathrm{Sp})$
- $\mathrm{Sh}(K)$ for K compact Hausdorff topological space

Some names

- Gaitzgory & Rozenblyum
- Hoyois, Safaronov, Scherotzke & Sibillia
- Clausen & Scholze
- Efimov
- Nikolaus & Krause
- Ramzi

Rigid Algebras

$$\text{Rigid categories} \subseteq \text{Symmetric monoidal categories}$$
$$\|$$
$$\mathsf{CAlg}(\mathrm{Pr}_{\mathrm{st}})$$

$\mathrm{Pr}_{\mathrm{st}} \rightsquigarrow$ Symmetric monoidal 2-category $\mathcal{U}$

$$\mathsf{CAlg}_{\mathsf{Rig}}(\mathcal{U}) \subseteq \mathsf{CAlg}(\mathcal{U})$$

- categories $=$ $(\infty, 1)$-categories
- 2-categories $=$ $(\infty, 2)$-categories

(2,2)-categories - Walter & Woods

Cospans

Cospans

$\mathcal{C}$ with finite colimits

$\mathsf{coSpan}(\mathcal{C})$ symmetric monoidal 2-category:

- objects: $A \in \mathcal{C}$
- 1-morpisms:

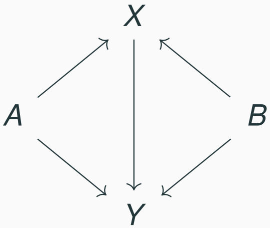
$$\begin{array}{ccccc} & & X & & \\ & \nearrow & & \nwarrow & \\ A & & & & B \end{array}$$

- composition:

$$\begin{array}{ccccccccc} & & & & X \sqcup_B Y & & & & \\ & & & \nearrow & \lrcorner & \nwarrow & & & \\ & & X & & & & Y & & \\ & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & \\ A & & & & B & & & & C \end{array}$$

Cospans

- 2-morpisms:



- Symmetric monoidal structure: $A \sqcup B$, $\emptyset$

  coproduct in $\mathcal{C}$, not in $\mathrm{coSpan}(\mathcal{C})$*!*

Rigid Algebras in Cospans

Every $A \in \mathcal{C}$ is canonically a rigid algebra in $\mathsf{coSpan}(\mathcal{C})$.

Main result: $\mathsf{coSpan}(\mathcal{C})$ is the free symmetric monoidal 2-category with this property.

$$\mathsf{Fun}^{\mathrm{rex}}(\mathcal{C}, \mathsf{CAlg}_{\mathsf{Rig}}(\mathcal{U})) \simeq \mathsf{Fun}^{\otimes,\mathrm{Rig}}(\mathsf{coSpan}(\mathcal{C}), \mathcal{U})$$

Adjunction and Duality

Left Adjoint

$\mathcal{U}$ a 2-category, $X, Y \in \mathcal{U}$.

$f \colon X \to Y$ is **left adjoint** if there exists:

$$f^R \colon Y \to X$$

$$c \colon f \circ f^R \to \mathrm{id}_Y \qquad\qquad u \colon \mathrm{id}_X \to f^R \circ f$$

$$\begin{array}{ccc} f & \xrightarrow{u} & f \circ f^R \circ f \\ & \searrow^{=} & \downarrow c \\ & & f \end{array} \qquad\qquad \begin{array}{ccc} f^R & \xrightarrow{u} & f^R \circ f \circ f^R \\ & \searrow^{=} & \downarrow c \\ & & f^R. \end{array}$$

In Cat - adjoint functors

Dualizable Object

$(\mathcal{U}, \otimes, \mathbb{1})$ symmetric monoidal category.

$X \in \mathcal{U}$ is **dualizable** if there exists:

$$X^\vee \in \mathcal{U}$$

$$\mathrm{ev}\colon X^\vee \otimes X \to \mathbb{1} \qquad \mathrm{coev}\colon \mathbb{1} \to X \otimes X^\vee$$

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{coev}\otimes\mathrm{id}} & X \otimes X^\vee \otimes X \\ & \searrow^{=} & \downarrow{\scriptstyle \mathrm{id}\otimes\mathrm{ev}} \\ & & X \end{array} \qquad \begin{array}{ccc} X^\vee & \xrightarrow{\mathrm{id}\otimes\mathrm{coev}} & X^\vee \otimes X \otimes X^\vee \\ & \searrow^{=} & \downarrow{\scriptstyle \mathrm{ev}\otimes\mathrm{id}} \\ & & X^\vee. \end{array}$$

In Vect_k - finite dimensional vector spaces

In Sp - finite spectra

Transpose

$(\mathcal{U}, \otimes, \mathbb{1})$ symmetric monoidal category, $X, Y \in \mathcal{U}$ dualizable.

$f\colon X \to Y$ has a **transpose** $f^t\colon Y^\vee \to X^\vee$

$$Y^\vee \xrightarrow{\mathrm{coev}_X} Y^\vee \otimes X \otimes X^\vee \xrightarrow{f} Y^\vee \otimes Y \otimes X^\vee \xrightarrow{\mathrm{ev}_Y} X^\vee$$

In Vect_k, this gives the dual map.

Adjunction and Duality in Cospans

$\mathcal{C}$ with finite colimits.

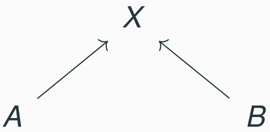
Every $A \in \mathcal{C}$ is self dual in $\mathsf{coSpan}(\mathcal{C})$.

$$\mathrm{ev}\colon \quad A \sqcup A \xrightarrow{\nabla} A \leftarrow \emptyset$$

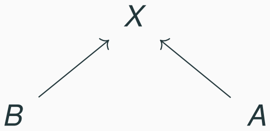
$$\mathrm{coev}\colon \quad \emptyset \rightarrow A \xleftarrow{\nabla} A \sqcup A$$

Transpose in Cospans

For any morphism in $\mathsf{coSpan}(\mathcal{C})$

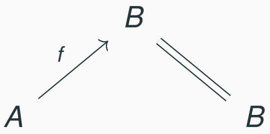


The transpose is the mirror image

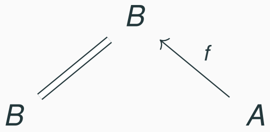


Left Adjoints in Cospans

Left adjoint morphisms in $\mathsf{coSpan}(\mathcal{C})$ are right way maps

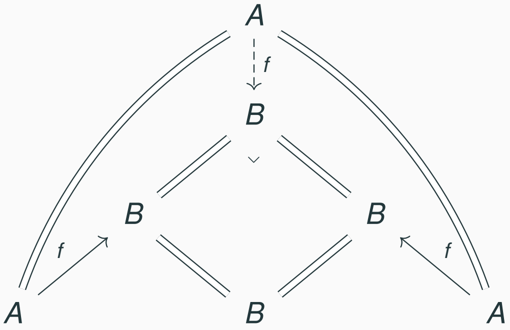


with right adjoint the transpose wrong way map



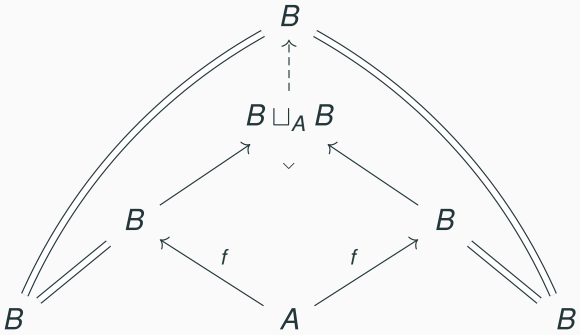
Left Adjoints in Cospans

Unit:



Left Adjoints in Cospans

Counit:



Rigid Algebras

Commutative Algebras

$(\mathcal{U}, \otimes, \mathbb{1})$ symmetric monoidal 2-category.

A **commutative algebra** $A \in \mathsf{CAlg}(\mathcal{U})$ consists of:

- unit $\eta \colon \mathbb{1} \to A$
- multiplication $\mu \colon A \otimes A \to A$
- compatibilities...

$A \in \mathsf{CAlg}(\mathcal{U})$ is **rigid** if:

1. $\eta \colon \mathbb{1} \to A$ is left adjoint
2. $\mu \colon A \otimes A \to A$ is left adjoint (in $\mathsf{BMod}_A(\mathcal{U})$)

$\mathsf{CAlg}_{\mathsf{Rig}}(\mathcal{U}) \subseteq \mathsf{CAlg}(\mathcal{U})$ full 2-subcategory.

Frobenius algebra

$(\mathcal{U}, \otimes, \mathbb{1})$ symmetric monoidal category.

A **commutative Frobenius algebra** is:

- $A \in \mathrm{CAlg}(\mathcal{U})$
- $\epsilon \colon A \to \mathbb{1}$
- $A \otimes A \xrightarrow{\mu} A \xrightarrow{\epsilon} \mathbb{1}$ exhibits A as self dual

A has a coalgebra structure:

- counit $\epsilon = \eta^t \colon A \to \mathbb{1}$
- comultiplication $\delta = \mu^t \colon A \to A \otimes A$

Frobenius algebras are equivalentely algebra + coalgebra + **Frobenius relations** $\leftsquigarrow$ δ is an A-bimodule map

Rigid is Frobenius

For $A \in \mathsf{CAlg}_{\mathsf{Rig}}(\mathcal{U})$, define:

- $\epsilon \colon A \to \mathbb{1}$ the right adjoint of η
- $\delta \colon A \to A \otimes A$ the right adjoint of μ

coalgebra + Frobenius relations $\Rightarrow$ A is Frobenius.

Rigid Algebras are a categorification of Frobenius algebras.

In 2-categories, the extra structure is canonical.

Adjoint is transpose

- η is left adjoint to $\epsilon = \eta^t$
- μ is left adjoint to $\delta = \mu^t$

Every $f\colon A \to B \in \mathsf{CAlg}_{\mathsf{Rig}}(\mathcal{U})$ is left adjoint to $f^t\colon B \to A$.

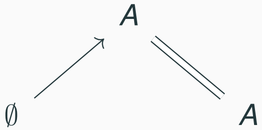
Every 2-morphism in $\mathsf{CAlg}_{\mathsf{Rig}}(\mathcal{U})$ is invertible.

$\mathsf{CAlg}_{\mathsf{Rig}}(\mathcal{U})$ is an $(\infty, 1)$-category.

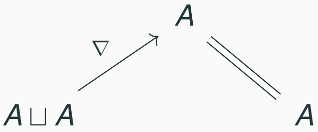
Rigid Algebras in Cospans

Every $A \in \mathcal{C}$ is canonically a rigid algebra in $\mathsf{coSpan}(\mathcal{C})$:

- unit



- multiplication



$\mathcal{C} \xrightarrow{\sim} \mathsf{CAlg}_{\mathsf{Rig}}(\mathsf{coSpan}(\mathcal{C}))$

Main Result

Setup

Categories of categories:

- Cat - categories
- $\mathsf{Cat}^{\mathrm{rex}}$ - categories with finite colimits
- $\mathsf{Cat}_2^{\otimes}$ - symmetric monoidal 2-categories

Functors:

- $\mathsf{coSpan} \colon \mathsf{Cat}^{\mathrm{rex}} \to \mathsf{Cat}_2^{\otimes}$
- $\mathsf{CAlg}_{\mathsf{Rig}} \colon \mathsf{Cat}_2^{\otimes} \to \mathsf{Cat}$

We want $\mathsf{CAlg}_{\mathsf{Rig}}$ to land in $\mathsf{Cat}^{\mathrm{rex}}$.

$\mathsf{CAlg}_{\mathsf{Rig}}(\mathcal{U})$ has initial object $\mathbb{1}$, what about pushouts?

Pushouts of Rigid Categories

$A, B, C \in \mathrm{CAlg}(\mathrm{Pr}_{\mathrm{st}})$ with maps $A \to B$ and $A \to C$.

$B \otimes_A C$ is pushout in $\mathrm{CAlg}(\mathrm{Pr}_{\mathrm{st}})$.

A, B, C rigid $\implies B \otimes_A C$ rigid.

Does this always work?

Not every $\mathcal{U} \in \mathrm{Cat}_2^{\otimes}$ has relative tensor products.

Restrict to subcategory $\mathrm{Cat}_2^{\mathrm{RigBar}} \subseteq \mathrm{Cat}_2^{\otimes}$.

Setup

$\mathsf{CAlg}_{\mathsf{Rig}} : \mathsf{Cat}_2^{\otimes} \to \mathsf{Cat}$ restricts to

$$\mathsf{CAlg}_{\mathsf{Rig}} : \mathsf{Cat}_2^{\mathrm{RigBar}} \to \mathsf{Cat}^{\mathrm{rex}} .$$

For every $\mathcal{C} \in \mathsf{Cat}^{\mathrm{rex}}$, $\mathsf{coSpan}(\mathcal{C}) \in \mathsf{Cat}_2^{\mathrm{RigBar}}$

$$\mathsf{coSpan} : \mathsf{Cat}^{\mathrm{rex}} \to \mathsf{Cat}_2^{\mathrm{RigBar}}$$

Theorem

There is an adjunction

$$\mathsf{coSpan}\colon\ \mathsf{Cat}^{\mathrm{rex}} \rightleftarrows \mathsf{Cat}_2^{\mathrm{RigBar}}\colon\ \mathsf{CAlg}_{\mathsf{Rig}}$$

$$\mathsf{Fun}^{\mathrm{rex}}(\mathcal{C}, \mathsf{CAlg}_{\mathsf{Rig}}(\mathcal{U})) \simeq \mathsf{Fun}^{\mathrm{RigBar}}(\mathsf{coSpan}(\mathcal{C}), \mathcal{U})$$

- unit $\mathcal{C} \xrightarrow{\sim} \mathsf{CAlg}_{\mathsf{Rig}}(\mathsf{coSpan}(\mathcal{C}))$
- counit $\mathsf{coSpan}(\mathsf{CAlg}_{\mathsf{Rig}}(\mathcal{U})) \to \mathcal{U}$
  from $\mathsf{CAlg}_{\mathsf{Rig}}(\mathcal{U}) \to \mathcal{U}$ (universal property of cospans).

Corollaries

Corollaries

1. $\mathsf{coSpan} : \mathsf{Cat}^{\mathrm{rex}} \to \mathsf{Cat}_2^{\mathrm{RigBar}}$ is fully faithful.
   $\mathsf{coSpan} : \mathsf{Cat}^{\mathrm{rex}} \to \mathsf{Cat}_2^{\otimes}$ is also fully faithful.

2. $\mathcal{S}^{\mathrm{fin}}$ the category of finite spaces.
   $\mathsf{coSpan}(\mathcal{S}^{\mathrm{fin}}) \in \mathsf{Cat}_2^{\mathrm{RigBar}}$ is free on a single rigid algebra.

$$\mathsf{Fun}^{\mathrm{RigBar}}(\mathsf{coSpan}(\mathcal{S}^{\mathrm{fin}}), \mathcal{U}) \simeq \mathsf{CAlg}_{\mathsf{Rig}}(\mathcal{U})$$

Six Functors Formalism

(Gaitzgory & Rozenblyum) A 6 functors formalism is a symmetric monoidal functor

$$\mathsf{coSpan}(\mathcal{C}, E) \to \mathrm{Pr}_{\mathrm{st}}.$$

Taking regular cospans, we get a (particular kind of) 6 functors formalism

$$\mathsf{coSpan}(\mathcal{C}) \to \mathrm{Pr}_{\mathrm{st}}.$$

Six Functors Formalism

$\mathcal{C} \to \mathsf{CAlg}(\mathrm{Pr}_{\mathrm{st}})$ that lands in rigid categories

$$\Updownarrow$$

$$\mathcal{C} \to \mathsf{CAlg}_{\mathsf{Rig}}(\mathrm{Pr}_{\mathrm{st}})$$

$$\Updownarrow$$

$$\mathsf{coSpan}(\mathcal{C}) \to \mathrm{Pr}_{\mathrm{st}}$$

$\mathsf{Sh}\colon \mathrm{CompHaus}^{\mathrm{op}} \to \mathsf{CAlg}(\mathrm{Pr}_{\mathrm{st}}) \rightsquigarrow$ 6 functors formalism on Sh

Thank You!